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RESEARCH ARTICLE

Semigroups corresponding to branches at infinity of coordinate lines in the affine plane

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Abstract We characterize in terms of characteristic sequences the semigroups corresponding to branches at infinity of plane affine curves Γ for which there exists a polynomial automorphism mapping Γ onto the axis $x = 0$.

Keywords Branch at infinity · Semigroup · Characteristic sequence · Polynomial automorphism · Abhyankar-Moh inequality

1 Introduction

Let $\mathbf{K}$ be an algebraically closed field of arbitrary characteristic and let $\gamma, \gamma', \ldots$ be plane algebroid branches centered at a point O of an algebraic nonsingular surface defined over $\mathbf{K}$. The semigroup $G(\gamma)$ of the branch γ is a subsemigroup of $\mathbf{N}$ consisting of 0 and all intersection numbers $i(\gamma, \gamma')$, where γ' varies over all algebroid curves not

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having γ as a component. We have $\min(G(\gamma)\backslash\{0\}) = \operatorname{ord}\gamma$ (the order (multiplicity) of the branch γ).
The semigroups of plane branches can be characterized in terms of sequences of generators. A sequence of positive integers $(r_0, \ldots, r_h)$ is said to be a *characteristic sequence* if it satisfies the following two axioms:

(**1**) Set $d_k = \gcd(r_0, \ldots, r_{k-1})$ for $1 \le k \le h+1$. Then $d_k > d_{k+1}$ for $1 \le k \le h$ and $d_{h+1} = 1$.
(**2**) $d_k r_k < d_{k+1} r_{k+1}$ for $1 \le k < h$.

We call r_0 the *initial term* of the characteristic sequence $(r_0, \ldots, r_h)$.
Let $G = r_0\mathbf{N} + \cdots + r_h\mathbf{N}$ be the semigroup generated by a characteristic sequence. Then $r_k = \min(G\backslash(r_0\mathbf{N} + \cdots + r_{k-1}\mathbf{N}))$ for $1 \le k \le h$ which shows that G and r_0 determine the sequence $(r_0, \ldots, r_h)$.

Bresinsky–Angermüller Semigroup Theorem

1. Let γ, λ be a pair of branches, where λ is nonsingular. Let $n = i(\gamma, \lambda) < +\infty$. Then the semigroup $G(\gamma)$ of the branch γ is generated by a characteristic sequence with initial term n.
2. Let $G \subseteq \mathbf{N}$ be a semigroup generated by a characteristic sequence with initial term $n > 0$. Then there exists a pair of branches γ, λ, where λ is a nonsingular branch such that $i(\gamma, \lambda) = n$ and $G(\gamma) = G$.

The above theorem was proved in [4] (for $\operatorname{char}\mathbf{K} = 0$), [2,6] (for arbitrary characteristic) for the transversal case: $i(\gamma, \lambda) = \operatorname{ord}\gamma$. A characteristic-blind proof of the theorem for arbitrary pairs γ, λ with $\lambda \neq \gamma$ nonsingular is given in [5].
It will be convenient to regard $\mathbf{K}^2$ as the projective plane $\mathbf{PK}^2$ without the line at infinity L. Let $\Gamma \subset \mathbf{K}^2$ be an affine irreducible curve. We say that Γ *has one branch at infinity* if its projective closure $\overline{\Gamma}$ intersects L at only one point O and $\overline{\Gamma}$ has only one branch centered at this point.
Let λ be the branch of the line at infinity L centered at O.
By the Bresinsky–Angermüller Theorem there exists a (unique) characteristic sequence $(r_0, \ldots, r_h)$ generating $G(\gamma)$ with initial term $r_0 = i(\gamma, \lambda) = \deg\Gamma$. We call $(r_0, \ldots, r_h)$ the *characteristic of* Γ *at infinity*.
The following result is of fundamental importance to studying the plane affine curves with one branch at infinity.

Abhyankar-Moh inequality
Assume that Γ is an affine irreducible curve of degree greater than 1 with one branch at infinity and let $(r_0, \ldots, r_h)$ be the characteristic of Γ at infinity. Suppose that $\gcd(\deg\Gamma, \operatorname{ord}\gamma) \not\equiv 0 \pmod{\operatorname{char}\mathbf{K}}$. Then

(**3**) $d_h r_h < r_0^2$.

The condition $\gcd(\deg\Gamma, \operatorname{ord}\gamma) \not\equiv 0 \pmod{\operatorname{char}\mathbf{K}}$ is automatically satisfied for $\operatorname{char}\mathbf{K} = 0$ and is essential if $\operatorname{char}\mathbf{K} \neq 0$. In [1] the Abhyankar-Moh inequality is formulated in terms of Laurent–Puiseux parametrizations of the branch γ (see also [7]). For the formulation given above we refer the reader to [3,9].

Conductor formula
Let Γ be an affine irreducible curve of degree greater than 1, rational, nonsingular with one branch at infinity. Let $(r_0, \ldots, r_h)$ be the characteristic of Γ at infinity. Then

(**4**) $\sum_{k=1}^{h}\left(\frac{d_k}{d_{k+1}} - 1\right) r_k = (r_0 - 1)^2.$

The Conductor Formula is a corollary to the genus formula for a plane curve in terms of its singularities. In [1] it is formulated in terms of Laurent–Puiseux parametrizations of the branch at infinity.
The aim of this note is to characterize the semigroups of nonnegative integers generated by the sequences satisfying the properties (**1**)–(**4**). Our main result is a counterpart of the Bresinsky–Angermüller Semigroup Theorem. We will not impose any restriction on the characteristic of **K**. The above quoted results gave motivation for writing this paper but will be not used in our proofs.

2 Result

A sequence of positive integers $(r_0, \ldots, r_h)$ will be called an Abhyankar-Moh characteristic sequence if it has properties (**1**)–(**4**) as in the Introduction. The following lemma is due to [3].

Lemma 2.1 (i) *Let* $(d_1, \ldots, d_{h+1})$ *be a sequence of integers such that* $d_1 > \cdots > d_{h+1} = 1$ *and* d_{k+1} *divides* d_k *for* $1 \le k \le h$. *Then the sequence* $(r_0, r_1, \ldots, r_h)$ *defined by* $r_0 = d_1$, $r_k = \frac{d_1^2}{d_k} - d_{k+1}$ *for* $1 \le k \le h$ *is an Abhyankar-Moh characteristic sequence with* $\gcd(r_0, \ldots, r_{k-1}) = d_k$ *for* $1 \le k \le h+1$.
(ii) *Let* $(r_0, r_1, \ldots, r_h)$ *be an Abhyankar-Moh characteristic sequence and let* $d_k = \gcd(r_0, \ldots, r_{k-1})$ *for* $1 \le k \le h+1$. *Then* $r_k = \frac{d_1^2}{d_k} - d_{k+1}$ *for* $1 \le k \le h$.

Proof A simple calculation gives the proof of (i). To check (ii) let $q_k = \frac{n^2}{d_k d_{k+1}} - \frac{r_k}{d_{k+1}}$ for $1 \le k \le h$. Then q_k is an integer and $q_k = \frac{n^2 - d_k r_k}{d_k d_{k+1}} \ge \frac{n^2 - d_h r_h}{d_k d_{k+1}} > 0$. Hence $q_k \ge 1$ and $\frac{n^2}{d_k} - r_k = d_{k+1} q_k \ge d_{k+1}$, which implies

$$\frac{n^2}{d_k} - d_{k+1} - r_k \ge 0 \quad \text{for } 1 \le k \le h. \tag{1}$$

On the other hand

$$\begin{aligned}
&\sum_{k=1}^{h}\left(\frac{d_k}{d_{k+1}}-1\right)\left(\frac{n^2}{d_k}-d_{k+1}-r_k\right)\\
&\quad=\sum_{k=1}^{h}\left(\frac{d_k}{d_{k+1}}-1\right)\left(\frac{n^2}{d_k}-d_{k+1}\right)-\sum_{k=1}^{h}\left(\frac{d_k}{d_{k+1}}-1\right)r_k\\
&\quad=(n-1)^2-(n-1)^2=0. \qquad (2)
\end{aligned}$$

Combining (1) and (2) we get $r_k = \frac{n^2}{d_k} - d_{k+1}$ for $1 \le k \le h$. $\square$

An affine curve $\Gamma \subset \mathbf{K}^2$ will be called a *coordinate line in the affine plane* (in short: *coordinate line*) if there exists a polynomial automorphism $(f, g) : \mathbf{K}^2 \longrightarrow \mathbf{K}^2$ such that $f = 0$ is the minimal equation of Γ.
Every coordinate line is an *embedded line* that is an affine curve biregular to an affine line $\mathbf{K}$ but the converse is not true if $\operatorname{char}\mathbf{K} \neq 0$ (see ([8]). Embedded lines are nonsingular, rational, with one branch at infinity.

Example 2.2 Let Γ be a graph of a polynomial in one variable of degree $n > 1$. Then Γ is a coordinate line. If γ is the unique branch at infinity of Γ then $G(\gamma) = n\mathbf{N}+(n-1)\mathbf{N}$.

The main result of this note is

Theorem 2.3 1. *Let Γ be a coordinate line of degree $n > 1$ with the branch at infinity γ. Then $G(\gamma)$ is generated by an Abhyankar-Moh characteristic sequence with initial term n.*
2. *Let $G \subseteq \mathbf{N}$ be a semigroup generated by an Abhyankar-Moh characteristic sequence with initial term $n > 1$. Then there exists a coordinate line Γ of degree n with the branch at infinity γ such that $G(\gamma) = G$.*

The proof of Theorem 2.3 is given in Sect. 3 of this note.

Remark 2.4 If $\operatorname{char}\mathbf{K} = 0$ then by the famous Abhyankar-Moh theorem every embedded line is a coordinate line. Determining the semigroups $G(\gamma)$ corresponding to branches γ of embedded lines remains an open question if $\operatorname{char}\mathbf{K} \neq 0$.

Example 2.5 (Semigroup in Nagata's example [8], p. 154) Let $\mathbf{K}$ be a field of characteristic $p > 0$ and let $a > 1$ be an integer coprime with p. Consider the polynomials $x(t) = t^{p^2}$, $y(t) = t + t^{ap}$. Then for $f(x, y) = (y^p - x^a)^p - x$ and $g(x, y) = y - (y^p - x^a)^a$ we have $f(x(t), y(t)) = 0$ and $g(x(t), y(t)) = t$ which shows that the affine curve Γ with equation $f(x, y) = 0$ is an embedded line.
We compute the semigroup of the branch at infinity γ of Γ. Let us distinguish two cases:
I. If $a < p$ then the Zariski closure of Γ intersects the line at infinity at $P = (1 : 0 : 0)$. We have $r_0 = \deg\Gamma = p^2$, $r_1 = \operatorname{ord}_P\bar{\Gamma} = p(p-a)$. Thus $d_1 = p^2$, $d_2 = \gcd(r_0, r_1) = p$ and $d_3 = 1$. Substituting these numbers to the conductor formula

$$\left(\frac{d_1}{d_2}-1\right)r_1+\left(\frac{d_2}{d_3}-1\right)r_2=(r_0-1)^2$$

we get $r_2 = p^3 + p(a-1) - 1$.
That is $G(\gamma) = p^2\mathbf{N} + p(p-a)\mathbf{N} + (p^3 + p(a-1) - 1)\mathbf{N}$.
II. If $a > p$ then the Zariski closure of Γ intersects the line at infinity at $Q = (0 : 1 : 0)$. We have $r_0 = \deg\Gamma = ap$, $r_1 = \mathrm{ord}_Q\bar{\Gamma} = p(a-p)$. Thus $d_1 = ap$, $d_2 = \gcd(r_0, r_1) = p$ and $d_3 = 1$. Substituting these numbers to the conductor formula we get $r_2 = a^2p + p(a-1) - 1$.
That is $G(\gamma) = ap\mathbf{N} + p(a-p)\mathbf{N} + (a^2p + p(a-1) - 1)\mathbf{N}$.
In both cases the semigroup $G(\gamma)$ satisfies properties (**1**), (**2**), (**4**) but not (**3**).

3 Proof

The following lemma is well-known.

Lemma 3.1 *Let $\gamma \neq \lambda$ be plane branches, where λ is nonsingular. Let $n = i(\gamma, \lambda)$. Suppose that there exist a characteristic sequence $(r_0, \ldots, r_h)$ with initial term $r_0 = n$ and a sequence of branches $(\gamma_1, \ldots, \gamma_{h+1})$, $\gamma_{h+1} = \gamma$ such that*

(1) $i(\gamma_k, \lambda) = \frac{n}{d_k}$ *for* $1 \leq k \leq h+1$,
(2) $i(\gamma_k, \gamma_{h+1}) = r_k$ *for* $1 \leq k \leq h$.

Then $G(\gamma) = r_0\mathbf{N} + \cdots + r_h\mathbf{N}$.

Proof See e.g. [5], Lemma 4.1. □

Let λ be a nonsingular branch. For any branches γ, γ' different from λ we put

$$d_\lambda(\gamma, \gamma') = \frac{i(\gamma, \gamma')}{i(\gamma, \lambda)i(\gamma', \lambda)}.$$

Lemma 3.2 *For any three branches $\gamma, \gamma', \gamma''$ at least two of the numbers $d_\lambda(\gamma, \gamma')$, $d_\lambda(\gamma, \gamma'')$, $d_\lambda(\gamma', \gamma'')$ are equal and the third one is not smaller than the other two.*

Proof See [5], Theorem 2.8. □

Proposition 3.3 *Let $(f_1, \ldots, f_{h+1})$ be a sequence of polynomials in $\mathbf{K}[x, y]$ and let $(n_1, \ldots, n_h)$ be a sequence of integers greater than 1 such that*

1. $1 = \deg f_1 < \ldots < \deg f_{h+1}$,
2. $(f_k, f_{k+1}) : \mathbf{K}^2 \longrightarrow \mathbf{K}^2$ *is a polynomial automorphism for* $1 \leq k \leq h$,
3. $\deg f_{k+1} = n_k \deg f_k$ *for* $1 \leq k \leq h$.

Let $d_k = n_k \cdots n_h$ for $1 \leq k \leq h$, $d_{h+1} = 1$ and let Γ be the affine curve with minimal equation $f_{h+1} = 0$, γ its branch at infinity. Then $G(\gamma) = r_0\mathbf{N} + \cdots + r_h\mathbf{N}$, where $r_0 = d_1$ and $r_k = \frac{d_1^2}{d_k} - d_{k+1}$ for $1 \leq k \leq h$.

Proof Let $\Gamma_k \subseteq \mathbf{K}^2$ be the affine curve with minimal equation $f_k = 0$ and let γ_k be the branch at infinity of Γ_k. In particular $\Gamma_{h+1} = \Gamma$ and $\gamma_{h+1} = \gamma$. All branches γ_k, $1 \leq k \leq h+1$ are centered at the common point at infinity O of the curves Γ_k. Let λ be the branch of the line at infinity L centered at O. Let $n = i(\gamma, \lambda)$. Observe that

$n = \deg \Gamma_{h+1} = n_1 \cdots n_h = d_1$ and $i(\gamma_k, \lambda) = \deg \Gamma_k = n_1 \cdots n_{k-1} = \frac{n}{d_k}$, that is the assumption (1) of Lemma 3.1 is satisfied.
Using Bézout's theorem to the curves $\overline{\Gamma}_k, \overline{\Gamma}_{k+1}$ which intersect in exactly one point in $\mathbf{K}^2$ we get

$$i(\gamma_k, \gamma_{k+1}) = \frac{n^2}{d_k d_{k+1}} - 1 \tag{3}$$

since the intersection in $\mathbf{K}^2$ is transversal. In particular $i(\gamma_h, \gamma_{h+1}) = \frac{n^2}{d_h d_{h+1}} - 1 = \frac{n^2}{d_h} - d_{h+1} = r_h$.
To check the assumption (2) of Lemma 3.1 we proceed by descendent induction on k. Assume that $i(\gamma_h, \gamma_{h+1}) = r_h, \cdots, i(\gamma_{k+1}, \gamma_{h+1}) = r_{k+1}$.
By inductive assumption $d_\lambda(\gamma_{k+1}, \gamma_{h+1}) = 1 - \frac{d_{k+1}d_{k+2}}{d_1^2}$ and by (3) $d_\lambda(\gamma_{k+1}, \gamma_k) = 1 - \frac{d_{k+1}d_k}{d_1^2}$.
Let us consider three branches $\gamma_k, \gamma_{k+1}, \gamma_h$. Since $d_\lambda(\gamma_{k+1}, \gamma_k) < d_\lambda(\gamma_{k+1}, \gamma_{h+1})$ we get by Lemma 3.2 applied to $\gamma_k, \gamma_{k+1}, \gamma_h$ that $d_\lambda(\gamma_k, \gamma_{h+1}) = d_\lambda(\gamma_{k+1}, \gamma_k)$ which implies $i(\gamma_k, \gamma_{h+1}) = \frac{d_1^2}{d_k}\left(1 - \frac{d_{k+1}d_k}{d_1^2}\right) = r_k$. □

Proposition 3.4 (Van der Kulk) *Let* $(f, g) : \mathbf{K}^2 \longrightarrow \mathbf{K}^2$ *be a polynomial automorphism. Then either* $\deg f$ *divides* $\deg g$ *or* $\deg g$ *divides* $\deg f$.

Proof See [10]. □

Lemma 3.5 *Let* $(g, f) : \mathbf{K}^2 \longrightarrow \mathbf{K}^2$ *be a polynomial automorphism. If* $\deg f > 1$ *then there exists* $\tilde{g}$ *such that* $(\tilde{g}, f) : \mathbf{K}^2 \longrightarrow \mathbf{K}^2$ *is a polynomial automorphism and* $\deg \tilde{g} < \deg f$.

Proof If $\deg g < \deg f$ then we put $\tilde{g} = g$. Suppose that $\deg g \geq \deg f$. By Proposition 3.4 $N = \frac{\deg g}{\deg f}$ is an integer. Each coordinate line has exactly one point at infinity. Since (g, f) is a non-linear automorphism the points at infinity of $g = 0$ and $f = 0$ coincide. Thus we can find a constant $c \in \mathbf{K}$ such that $\deg(g - cf^N) < \deg g$ (cf. [10], p. 37). Replace g by $g - cf^N$. Repeating this procedure a finite number of times we get a polynomial automorphism $(\tilde{g}, f) : \mathbf{K}^2 \longrightarrow \mathbf{K}^2$ such that $\deg \tilde{g} < \deg f$. □

Proof of Theorem 2.3 (i) Let Γ be a coordinate line with the minimal equation $f = 0$ of degree $n > 1$. Let γ be the branch at infinity of Γ.
Using Lemma 3.5 we construct a sequence of polynomials $(f_1, \ldots, f_{h+1})$, where $f_{h+1} = f$ such that $(f_k, f_{k+1}) : \mathbf{K}^2 \longrightarrow \mathbf{K}^2$ is a polynomial automorphism for $1 \leq k \leq h$ and $\deg f_k < \deg f_{k+1}$. By Proposition 3.4 $\deg f_k$ divides $\deg f_{k+1}$. Let $n_k = \frac{\deg f_{k+1}}{\deg f_k}$ for $1 \leq k \leq h$.
Applying Proposition 3.3 to the sequences $(f_1, \ldots, f_{h+1})$ and $(n_1, \ldots, n_h)$ we get $G(\gamma) = r_0\mathbf{N} + \cdots + r_h\mathbf{N}$, where $r_0 = n$ and $r_k = \frac{n^2}{d_k} - d_{k+1}$ for $1 \leq k \leq h$. The sequence $(r_0, \ldots, r_h)$ is an Abhyankar-Moh sequence by Lemma 2.1 (i).
(ii) Let $G \subseteq \mathbf{N}$ be a semigroup generated by an Abhyankar-Moh sequence $(r_0, \ldots, r_h)$ with the initial term $r_0 = n > 1$. Let $d_k = \gcd(r_0, \ldots, r_{k-1})$ for $1 \leq k \leq h+1$. Then $r_k = \frac{n^2}{d_k} - d_{k+1}$ by Lemma 2.1 (ii). Let $n_k = \frac{d_k}{d_{k+1}}$ for $1 \leq k \leq h+1$.

Set

$$
\begin{aligned}
f_1 &= y, \\
f_2 &= y^{n_1} - x, \\
f_{k+1} &= f_k^{n_k} - f_{k-1} \quad \text{for } 2 \leq k \leq h.
\end{aligned}
$$

Then the sequences $(f_1, \ldots, f_{h+1})$ and $(n_1, \ldots, n_h)$ satisfy the assumptions of Proposition 3.3 and it suffices to take Γ as the plane affine curve with minimal equation $f_{h+1} = 0$. □

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