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FORMAL AND CONVERGENT SOLUTIONS OF ANALYTIC EQUATIONS

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Impressed by the power of the Preparation Theorem – indeed, it prepares us so well! – I considered "Weierstrass Preparation Theorem and its immediate consequences" as a possible title for the entire book.

Sheeram S. Abhyankar, Preface to [1]

Abstract. We provide the detailed proof of a sharpened version of the M. Artin Approximation Theorem.

1. Introduction

The famous Approximation Theorem of M. Artin [2] asserts that any formal solution of a system of analytic equations can be approximated by convergent solutions up to a given order. In my PhD thesis [7] I was able by analysis of the argument used in [2] to sharpen the Approximation Theorem: any formal solution can be obtained by specializing parameters in a convergent parametric solution. The theorem was announced with a sketch of proof in [8]. The aim of theses notes is to present the detailed proof of this result. It is based on the Weierstrass Preparation Theorem. The other tools are: a Jacobian Lemma which is an elementary version of the Regularity Jacobian Criterion used in [2], the trick of Kronecker (introducing and specializing variables) and a generalization of the Implicit Function Theorem due to Bourbaki [4] and Tougeron [10]. All theses ingredients are vital in the proofs of some other results of this type (see [3], [11]).

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For more information on approximation theorems in local analytic geometry we refer the reader to Teissier's article [9] and to Chapter 8 of the book [5].

Let $\mathbb{K}$ be a field of characteristic zero with a non-trivial valuation. We put $\mathbb{K}[[x]] = \mathbb{K}[[x_1, \dots, x_n]]$ the ring of formal power series in variables $x = (x_1, \dots, x_n)$ with coefficients in $\mathbb{K}$. If $f = \sum_{k \geq p} f_k$ is a nonzero power series represented as the sum of homogeneous forms with $f_p \neq 0$ then we write $\operatorname{ord} f = p$. Additionally we put $\operatorname{ord} 0 = +\infty$ and use the usual conventions on the symbol $+\infty$. The constant term of any series $f \in \mathbb{K}[[x]]$ we denote by $f(0)$. A power series $u \in \mathbb{K}[[x]]$ is a unit if $uv = 1$ for a power series $v \in \mathbb{K}[[x]]$. Note that u is a unit if and only if $u(0) \neq 0$. The non-units of $\mathbb{K}[[x]]$ form the unique maximal ideal $\mathbf{m_x}$ of the ring $\mathbb{K}[[x]]$. The ideal $\mathbf{m_x}$ is generated by the variables $x_1, \dots, x_n$. One has $f \in \mathbf{m_x}^c$, where $c > 0$ is an integer, if and only if $\operatorname{ord} f \geq c$. Recall that if $g_1, \dots, g_n \in \mathbb{K}[[y]]$, $y = (y_1, \dots, y_n)$ are without constant term then the series $f(g_1, \dots, g_n) \in \mathbb{K}[[y]]$ is well-defined. The mapping which associates with $f \in \mathbb{K}[[x]]$ the power series $f(g_1, \dots, g_n)$ is the unique homomorphism sending x_i for g_i for $i = 1, \dots, n$. Let $\mathbb{K}\{x\}$ be the subring of $\mathbb{K}[[x]]$ of all convergent power series. Then $\mathbb{K}\{x\}$ is a local ring. If $g_1, \dots, g_n \in \mathbb{K}\{y\}$ then $f(g_1, \dots, g_n) \in \mathbb{K}\{y\}$ for any $f \in \mathbb{K}\{x\}$.

In what follows we use intensively the Weierstrass Preparation and Division Theorems. The reader will find the basic facts concerning the rings of formal and convergent power series in [1], [6] and [12].

Let $f(x, y) = (f_1(x, y), \dots, f_m(x, y)) \in \mathbb{K}\{x, y\}^m$ be convergent power series in the variables $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_N)$ where m, n, N are arbitrary non-negative integers. The theorem quoted below is the main result of [2].

The Artin Approximation Theorem. *Suppose that there exists a sequence of formal power series $\bar{y}(x) = (\bar{y}_1(x), \dots, \bar{y}_N(x))$ without constant term such that*

$$f(x, \bar{y}(x)) = 0 \ .$$

Then for any integer $c > 0$ there exists a sequence of convergent power series $y(x) = (y_1(x), \dots, y_N(x))$ such that

$$f(x, y(x)) = 0 \text{ and } y(x) \equiv \bar{y}(x) \ (\operatorname{mod} \mathbf{m_x}^c) \ .$$

The congruence condition means that the power series $y_\nu(x) - \bar{y}_\nu(x)$ are of order $\geq c$ i.e. the coefficients of monomials of degree $< c$ agree in $y_\nu(x)$ and $\bar{y}_\nu(x)$. We will deduce the Artin Approximation Theorem from the following result stated with a sketch of proof in [8].

Theorem. *With the notation and assumptions of the Artin theorem there exists a sequence of convergent power series $y(x, t) = (y_1(x, t), \dots, y_N(x, t)) \in \mathbb{K}\{x, t\}^N$, $y(0, 0) = 0$, where $t = (t_1, \dots, t_S)$ are new variables, $S \geq 0$, and a sequence of formal power series $\bar{t}(x) = (\bar{t}_1(x), \dots, \bar{t}_S(x)) \in \mathbb{K}[[x]]^S$, $\bar{t}(0) = 0$ such that*

$$f(x, y(x, t)) = 0 \text{ and } \bar{y}(x) = y(x, \bar{t}(x)) \ .$$

The construction of the parametric solution $y(x,t)$ depends on the given formal solution $\bar{y}(x)$. To get the Artin Approximation Theorem from the stated above result fix an integer $c > 0$. Let $y(x,t)$ and $\bar{t}(x)$ be series such as in the theorem and let $t(x) = (t_1(x), \dots, t_S(x)) \in \mathbb{K}\{x\}^S$ be convergent power series such that $t(x) \equiv \bar{t}(x) \bmod \mathbf{m}_{\mathbf{x}}^c$. Therefore $y(x, t(x)) \equiv y(x, \bar{t}(x)) \bmod \mathbf{m}_{\mathbf{x}}^c$ and it suffices to set $y(x) = y(x, t(x))$. □

Before beginning the proof of the theorem let us indicate two corollaries of it.

Corollary 1. *Assume that $m = N$, $f(x, \bar{y}(x)) = 0$ and*

$$\det \frac{J(f_1, \dots, f_N)}{J(y_1, \dots, y_N)}(x, \bar{y}(x)) \neq 0 .$$

Then the power series $\bar{y}(x)$ are convergent.

Proof. Let $y(x,t)$ and $\bar{t}(x)$ be power series without constant term such that $f(x, y(x,t)) = 0$ and $\bar{y}(x) = y(x, \bar{t}(x))$. It is easy to check by differentiation of equalities $f(x, y(x,t)) = 0$ that $(\partial y_\nu / \partial t_\sigma)(x,t) = 0$ for $\nu = 1, \dots, N$ and $\sigma = 1, \dots, S$. Therefore the series $y(x,t)$ are independent of t and the series $\bar{y}(x)$ are convergent. □

Corollary 2. *If $f(x,y) \in \mathbb{K}\{x,y\}$ is a nonzero power series of $n+1$ variables $(x,y) = (x_1, \dots, x_n, y)$ and $\bar{y}(x)$ is a formal power series without constant term such that $f(x, \bar{y}(x)) = 0$ then $\bar{y}(x)$ is a convergent power series.*

Proof. By Corollary 1 it suffices to check that there exists a power series $g(x,y) \in \mathbb{K}\{x,y\}$ such that $g(x, \bar{y}(x)) = 0$ and $(\partial g / \partial y)(x, \bar{y}(x)) \neq 0$. Let $I = \{g(x,y) \in \mathbb{K}\{x,y\} : g(x, \bar{y}(x)) = 0\}$. Then $I \neq \mathbb{K}\{x,y\}$ is a prime ideal of $\mathbb{K}\{x,y\}$. Assume the contrary, that is, that for every $g \in I$: $(\partial g / \partial y) \in I$. Then we get by differentiating the equality $g(x, \bar{y}(x)) = 0$ that $(\partial g / \partial x_i) \in I$ for $i = 1, \dots, n$ and, by induction, all partial derivatives of g lie in I. Consequently $g = 0$ for every $g \in I$ i.e. $I = (0)$. A contradiction since $0 \neq f \in I$. □

2. Reduction to the case of simple solutions

We keep the notation introduced in Introduction. We will call a sequence of formal power series $\bar{y}(x) \in \mathbb{K}[[x]]$, $\bar{y}(0) = 0$ a *simple solution* of the system of analytic equations $f(x,y) = 0$ if $f(x, \bar{y}(x)) = 0$ and

$$\operatorname{rank} \frac{J(f_1, \dots, f_m)}{J(y_1, \dots, y_N)}(x, \bar{y}(x)) = m .$$

Thus, in this case, $m \leq N$.

In what follows we need

The Jacobian Lemma. *Let I be a nonzero prime ideal of the ring $\mathbb{K}\{x\}$, $x = (x_1, \dots, x_n)$. Then there exist an integer r: $1 \leq r \leq n$ and covergent power series $h_1, \dots, h_r \in I$ such that*

(i) $\operatorname{rank}\dfrac{J(h_1,\dots,h_r)}{J(x_1,\dots,x_n)}(\operatorname{mod} I) = r$,

(ii) $\forall h \in I$, $\exists a \notin I$ *such that* $ah \in (h_1,\dots,h_r)\mathbb{K}\{x\}$.

Before proving the above lemma let us note that it is invariant with respect to $\mathbb{K}$-linear nonsingular transformations. If Φ is an authomorphism of $\mathbb{K}\{x\}$ defined by

$$\Phi(f(x_1,\dots,x_n)) = f\left(\sum_{j=1}^{n} c_{1j}x_j,\dots,\sum_{j=1}^{n} c_{nj}x_j\right)$$

with $\det(c_{ij}) \neq 0$ then the Jacobian Lemma is true for I if and only if it is true for $\Phi(I)$.

Proof of the Jacobian Lemma (by induction on the number n of variables x_i). If $n = 1$ then $I = (x_1)\mathbb{K}\{x_1\}$ and $h_1 = x_1$. Suppose that $n > 1$ and that the lemma is true for prime ideals of the ring of power series in $n-1$ variables. Using a $\mathbb{K}$-linear nonsingular transformation we may assume that the ideal I contains a power series x_n-regular of order $k > 0$ i.e. such that the term x_n^k appears in the power series with a non-zero coefficients. Therefore, by the Weierstrass Preparation Theorem I contains a distinguished polynomial

$$w(x',x_n) = x_n^k + a_1(x')x_n^{k-1} + \cdots + a_k(x'),\ \text{where } x' = (x_1,\dots,x_{n-1}) .$$

By the Weierstrass Division Theorem every power series $h = h(x)$ is of the form $h(x) = q(x)w(x',x_n)+r(x',x_n)$ where $r(x',x_n)$ is an x_n-polynomial (of degree $< k$). Therefore, the ideal I is generated by the power series which are polynomials in x_n and to prove the Jacobian Lemma it suffices to find power series $h_1,\dots,h_r$ such that (i) holds and (ii) is satisfied for $h \in I \cap \mathbb{K}\{x'\}[x_n]$.

Let $I' = I \cap \mathbb{K}\{x'\}$ and consider the set $I \setminus I'[x_n]$. Clearly $w(x',x_n) \in I \setminus I'[x_n]$. Let

$$h_1(x',x_n) = c_0(x')x_n^l + c_1(x')x_n^{l-1} + \cdots + c_l(x')$$

be a polynomial in x_n of the minimal degree l, $l \geq 0$, which belongs to $I \setminus I'[x_n]$. Since the degree $l \geq 0$ is minimal, we have

$$\begin{aligned} l &> 0 ,\\ c_0(x') &\notin I' ,\\ \tfrac{\partial h_1}{\partial x_n} &\in I . \end{aligned}$$

Let $h(x',x_n) \in I$ be a polynomial in x_n. Dividing $h(x',x_n)$ by $h_1(x',x_n)$ (Euklid's division) we get

(E) $c_0(x')^p\, h(x',x_n) = q(x',x_n)\, h_1(x',x_n) + r_1(x',x_n)$,

where x_n-degree of $r_1(x',x_n)$ is less than l and $p \geq 0$ is an integer. Since the x_n-degree of $r_1(x',x_n)$ is $< l$ then all coefficients of $r_1(x',x_n)$ lie in I'. If $I' = (0)$ then $r_1(x',x_n) = 0$ and (E) proves the Jacobian Lemma.

If $I' \neq (0)$ then by the induction hypothesis there exists series $h_2, \dots, h_r \in I'$ such that

(i′) $\operatorname{rank} \dfrac{J(h_2, \dots, h_r)}{J(x_1, \dots, x_{n-1})} (\operatorname{mod} I') = r - 1$,
(ii′) $\forall h' \in I'$, $\exists a' \notin I'$ such that $a'h' \in (h_2, \dots, h_r)\mathbb{K}\{x'\}$.

We claim that $h_1, \dots, h_r$ satisfy (i) and (ii) of the Jacobian Lemma. To check (i) observe that

$$\det \frac{J(h_1, \dots, h_r)}{J(x_{i_1}, \dots, x_{i_{r-1}}, x_n)} = \det \frac{J(h_2, \dots, h_r)}{J(x_{i_1}, \dots, x_{i_{r-1}})} \cdot \frac{\partial h_1}{\partial x_n} ,$$

where $i_1, \dots, i_{r-1} \in \{1, \dots, n-1\}$ and use (i′). Applying (ii′) to the coefficients of $r_1(x', x_n)$ we find a power series $a'(x')$ such that $a'(x')\, r_1(x', x_n) \in (h_2, \dots, h_r)\mathbb{K}\{x\}$. By (E) we get $a(x')\, h(x', x_n) \in (h_1, \dots, h_r)\mathbb{K}\{x\}$ where $a(x') = a'(x')\, c_0(x')^p \notin I$ which proves (ii). □

Now, we can check

Proposition 2.1. *Let* $f(x,y) = (f_1(x,y), \dots, f_m(x,y)) \in \mathbb{K}\{x,y\}^m$, $f(x,y) \neq 0$, $\bar{y}(x) = (\bar{y}_1(x), \dots, \bar{y}_N(x)) \in \mathbb{K}[[x]]$, $\bar{y}(0) = 0$, *be formal power series such that* $f(x, \bar{y}(x)) = 0$. *Then there exist convergent power series* $h(x,y) = (h_1(x,y), \dots, h_r(x,y)) \in \mathbb{K}\{x,y\}^r$ *such that*

(i) $h(x, \bar{y}(x)) = 0$,
(ii) $\operatorname{rank} \dfrac{J(h_1, \dots, h_r)}{J(y_1, \dots, y_N)} (x, \bar{y}(x)) = r$,
(iii) *suppose that there exist formal power series* $y(x,t) = (y_1(x,t), \dots, y_N(x,t))$, $y(0,0) = 0$ *and* $\bar{t}(x) = (\bar{t}_1(x), \dots, \bar{t}_S(x))$, $\bar{t}(0) = 0$, *such that* $h(x, y(x,t)) = 0$ *and* $\bar{y}(x) = y(x, \bar{t}(x))$. *Then* $f(x, y(x,t)) = 0$.

Proof. Consider the prime ideal

$$I = \{g(x,y) \in \mathbb{K}\{x,y\} : \; g(x, \bar{y}(x)) = 0\} .$$

Clearly $f_1(x,y), \dots, f_m(x,y) \in I$ and $I \neq (0)$. By the Jacobian Lemma there exist formal power series $h_1(x,y), \dots, h_r(x,y) \in I$ such that

- $\operatorname{rank} \dfrac{J(h_1, \dots\dots\dots, h_r)}{J(x_1, \dots, x_n, y_1, \dots, y_N)} (x, \bar{y}(x)) = r$,
- $\forall g \in I$, $\exists a \notin I$ such that $a(x,y)\, g(x,y) \in (h_1, \dots, h_r)\mathbb{K}\{x,y\}$.

We claim that $h_1, \dots, h_r$ satisfy the conditions (i), (ii), (iii). Condition (i) holds since $h_1, \dots, h_r \in I$. To check (ii) it suffices to observe that

(J) $\operatorname{rank} \dfrac{J(h_1, \dots\dots\dots, h_r)}{J(x_1, \dots, x_n, y_1, \dots, y_N)} (x, \bar{y}(x)) = \operatorname{rank} \dfrac{J(h_1, \dots, h_r)}{J(y_1, \dots, y_N)} (x, \bar{y}(x))$.

Indeed, differentiating the equations $h_i(x,\bar{y}(x)) = 0$, $i = 1,\dots,r$, we get

$$\frac{\partial h_i}{\partial x_j}(x,\bar{y}(x)) + \sum_{\nu=1}^{N} \frac{\partial h_i}{\partial y_\nu}(x,\bar{y}(x))\frac{\partial \bar{y}_\nu}{\partial x_j} = 0 \text{ for } j = 1,\dots,n$$

and (J) follows. To check (iii) let us write

$$a_i(x,y) f_i(x,y) = \sum_{k=1}^{r} a_{i,k}(x,y) h_k(x,y) \text{ in } \mathbb{K}\{x,y\}\ ,$$

where $a_i(x,y) \notin I$ for $i = 1,\dots,m$. Thus $a_i(x,\bar{y}(x)) \neq 0$ and $a_i(x,y(x,t)) \neq 0$ since $\bar{y}(x) = y(x,\bar{t}(x))$ and (iii) follows. $\square$

3. The Bourbaki–Tougeron implicit function theorem

Let $f(x,y) = (f_1(x,y),\dots,f_m(x,y)) \in \mathbb{K}\{x,y\}^m$ be convergent power series in variables $x = (x_1,\dots,x_n)$ and $y = (y_1,\dots,y_N)$. Suppose that $m \le N$ and put

$$J(x,y) = \frac{J(f_1,\dots,f_m)}{J(y_{N-m+1},\dots,y_N)} \text{ and } \delta(x,y) = \det J(x,y)\ .$$

Let $M(x,y)$ be the adjoint of the matrix $J(x,y)$. Thus we have

$$M(x,y)J(x,y) = J(x,y)M(x,y) = \delta(x,y) I_m$$

where I_m is the identity matrix of m rows and m columns. Let $g(x,y) = (g_1(x,y), \dots, g_m(x,y)) \in \mathbb{K}\{x,y\}^m$ be convergent power series defined by

$$\begin{bmatrix} g_1(x,y) \\ \vdots \\ g_m(x,y) \end{bmatrix} = M(x,y) \begin{bmatrix} f_1(x,y) \\ \vdots \\ f_m(x,y) \end{bmatrix}\ .$$

It is easy to see that

(a) $g_i(x,y) \in (f_1(x,y),\dots,f_m(x,y))\mathbb{K}\{x,y\}$ for $i = 1,\dots,m$

and

(b) $\delta(x,y) f_i(x,y) \in (g_1(x,y),\dots,g_m(x,y))\mathbb{K}\{x,y\}$ for $i = 1,\dots,m$.

Now, we can state

The Bourbaki-Tougeron implicit function theorem. *Suppose that there exists a sequence of formal power series $y^0(x) = (y_1^0(x),\dots,y_N^0(x))$, $y^0(0) = 0$, such that*

$$g_i(x,y^0(x)) \equiv 0 \bmod \delta(x,y^0(x))^2 \mathbf{m_x} \quad \textit{for } i = 1,\dots,m\ .$$

Then

I. *Let* $y_\nu(x,t) = y^0_\nu(x) + \delta(x,y^0(x))^2 t_\nu$ *for* $\nu = 1,\dots,N-m$ *where* $t = (t_1,\dots,t_{N-m})$ *are new variables. Then there exists a unique sequence of formal power series* $u(x,t) = (u_{N-m+1}(x,t),\dots,u_N(x,t)) \in \mathbb{K}[[x,t]]^m$, $u(0,0) = 0$, *such that if we let* $y_\nu(x,t) = y^0_\nu(x) + \delta(x,y^0(x))u_\nu(x,t)$ *for* $\nu = N-m+1,\dots,N$ *and* $y(x,t) = (y_1(x,t),\dots,y_N(x,t))$ *then*

$$f(x,y(x,t)) = 0 \text{ in } \mathbb{K}[[x,t]]\,.$$

If the series $y^0(x)$ *are convergent then* $u(x,t)$ *and* $y(x,t)$ *are covergent as well.*

II. *For every sequence of formal power series* $\bar y(x) = (\bar y_1(x),\dots,\bar y_N(x))$, $\bar y(0) = 0$, *the following two conditions are equivalent*
  (i) *there exists a sequence of formal power series* $\bar t(x) = (\bar t_1(x),\dots,\bar t_{N-m}(x))$, $\bar t(0) = 0$, *such that* $\bar y(x) = y(x,\bar t(x))$,
  (ii) $f(x,\bar y(x)) = 0$ *and*

$$\bar y_\nu(x) \equiv y^0_\nu(x) \bmod \delta(x,y^0(x))^2\mathbf{m_x} \text{ for } \nu = 1,\dots,N-m$$
$$\bar y_\nu(x) \equiv y^0_\nu(x) \bmod \delta(x,y^0(x))\mathbf{m_x} \text{ for } \nu = N-m+1,\dots,N\,.$$

Remark. In what follows we call

$y^0(x)$	an approximate solution of the system $f(x,y) = 0$,
$y(x,t)$	a parametric solution determined by the approximate solution $y^0(x)$
$\bar y(x)$	satifying (i) or (ii) a subordinate solution to the approximate solution $y^0(x)$

Proof. Let $v = (v_1,\dots,v_N)$ and $h = (h_1,\dots,h_n)$ be variables. Taylor's formula reads

$$\text{(T)}\quad \begin{aligned}\begin{bmatrix} f_1(x,v+h)\\ \vdots \\ f_m(x,v+h)\end{bmatrix} &= \begin{bmatrix} f_1(x,v)\\ \vdots \\ f_m(x,v)\end{bmatrix} + \frac{J(f_1,\dots,f_m)}{J(y_1,\dots,y_{N-m})}(x,v)\begin{bmatrix} h_1\\ \vdots \\ h_{N-m}\end{bmatrix}\\ &\quad + J(x,v)\begin{bmatrix} h_{N-m+1}\\ \vdots \\ h_N\end{bmatrix} + \begin{bmatrix} P_1(u,v,h)\\ \vdots \\ P_m(u,v,h)\end{bmatrix}\end{aligned}$$

where $P_i(x,v,h) \in (h_1,\dots,h_N)^2\mathbb{K}\{x,v,h\}$ for $i = 1,\dots,m$. Let $u = (u_{N-m+1},\dots,u_N)$ be variables and put

$$F_i(x,t,u) = f_i(x,y_1(x,t),\dots,y_{N-m}(x,t),y^0_{N-m+1}(x) + \delta(x,y^0(x))u_{N-m+1},\\ \dots,y^0_N(x) + \delta(x,y^0(x))u_N)\,.$$

Substituting in Taylor's formula (T) $v_i = y_i^0(x)$ for $i = 1, \ldots, N$, $h_i = \delta(x, y^0(x))^2 t_i$ for $i = 1, \ldots, N-m$ and $h_i = \delta(x, y^0(x)) u_i$ for $i = N-m+1, \ldots, N$ we get

$$\begin{bmatrix} F_1(x,t,u) \\ \vdots \\ F_m(x,t,u) \end{bmatrix} = \begin{bmatrix} f_1(x,y^0(x)) \\ \vdots \\ f_m(x,y^0(x)) \end{bmatrix} + \delta(x,y^0(x))^2 \frac{J(f_1,\ldots,f_m)}{J(y_1,\ldots,y_{N-m})}(x,y^0(x)) \begin{bmatrix} t_1 \\ \vdots \\ t_{N-m} \end{bmatrix}$$
$$+ \delta(x,y^0(x)) J(x,y^0(x)) \begin{bmatrix} u_{N-m+1} \\ \vdots \\ u_N \end{bmatrix} + \delta(x,y^0(x))^2 \begin{bmatrix} Q_1(x,t,u) \\ \vdots \\ Q_m(x,t,u) \end{bmatrix}$$

where $Q_i(x,t,u) \in (t,u)^2 \mathbb{K}\{x,t,u\}$ for $i = 1, \ldots, m$. Multiplying the above identity by the matrix $M(x, y^0(x))$ and taking into account that $M(x,y^0(x))J(x,y^0(x)) = \delta(x,y^0(x)) I_m$ and $g_i(x,y^0(x)) \equiv 0 \pmod{\delta(x,y_0(x))^2 \mathbf{m_x}}$ for $i = 1, \ldots, m$, we get

$$(*) \qquad M(x,y^0(x)) \begin{bmatrix} F_1(x,t,u) \\ \vdots \\ F_m(x,t,u) \end{bmatrix} = \delta(x,y^0(x))^2 \begin{bmatrix} G_1(x,t,u) \\ \vdots \\ G_m(x,t,u) \end{bmatrix}$$

where $G_i(0,0,0) = 0$ for $i = 1, \ldots, m$. Differentiating (*) we obtain

$$M(x,y^0(x)) \frac{J(F_1,\ldots,F_m)}{J(u_{N-m+1},\ldots,u_N)}(x,t,u) = \delta(x,y^0(x))^2 \frac{J(G_1,\ldots,G_m)}{J(u_{N-m+1},\ldots,u_N)}(x,t,u)$$

which implies

$$(**) \quad \det \frac{J(F_1,\ldots,F_m)}{J(u_{N-m+1},\ldots,u_N)}(x,t,u)$$
$$= \delta(x,y^0(x))^{m+1} det \frac{J(G_1,\ldots,G_m)}{J(u_{N-m+1},\ldots,u_N)}(x,t,u)$$

since $\det M(x,y^0(x)) = \delta(x,y^0(x))^{m-1}$. On the other hand

$$\frac{J(F_1,\ldots,F_m)}{J(u_{N-m+1},\ldots,u_N)}(x,0,0) = \delta(x,y^0(x)) J(x,y^0(x))$$

and

$$\det \frac{J(F_1,\ldots,F_m)}{J(u_{N-m+1},\ldots,u_N)}(x,0,0) = \delta(x,y^0(x))^{m+1} \, .$$

Therefore we get from (**)

$$\det \frac{J(G_1,\ldots,G_m)}{J(u_{N-m+1},\ldots,u_N)}(x,0,0) = 1 \, ,$$

in particular

$$\det \frac{J(G_1,\ldots,G_m)}{J(u_{N-m+1},\ldots,u_N)}(0,0,0) = 1 \, .$$

By the Implicit Function Theorem there exist formal power series

$$u(x,t) = (u_{N-m+1}(x,t), \ldots, u_N(x,t))$$

such that

$$(G_1(x,t,u),\dots,G_m(x,t,u))\mathbb{K}[[x,t,u]] \\ = (u_{N-m+1}-u_{N-m+1}(x,t),\dots,u_N-u_N(x,t))\mathbb{K}[[x,t,u]]\ .$$

If $y^0(x)$ are convergent then $G(x,t,u)$ and $u(x,t)$ are convergent as well. In particular $G(x,t,u(x,t))=0$ and by (*) $F(x,t,u(x,t))=0$ which implies $f(x,y(x,t))=0$ where $y_\nu(x,t) = y^0_\nu(x)+\delta(x,y^0(x))u_\nu(x,t)$ for $\nu = N-m+1,\dots,N$. Let $\tilde{u}(x,t) = (\tilde{u}_{N-m+1}(x,t),\dots,\tilde{u}_N(x,t))$, $\tilde{u}(0,0)=0$, be power series such that $f(x,\tilde{y}(x,t))=0$ where

$$\tilde{y}(x,t) = (y_1(x,t),\dots,y_{N-m}(x,t),y^0_{N-m+1}(x)+\delta(x,y^0(x))\tilde{u}_{N-m+1}(x,t), \\ \dots,y^0_N(x)+\delta(x,y^0(x))\tilde{u}_N(x,t))\ .$$

Then $F(x,t,\tilde{u}(x,t))=0$ and by (*) $G(x,t,\tilde{u}(x,t))=0$. Thus we get $\tilde{u}(x,t) = u(x,t)$. This proves the first part of the Bourbaki-Tougeron Implicit Function Theorem. To check the second part it suffices to observe that for any formal power series $\bar{t}(x) = (\bar{t}_1(x),\dots,\bar{t}_{N-m}(x))$ and $\bar{u}(x) = (\bar{u}_{N-m+1}(x),\dots,\bar{u}_N(x))$ without constant term $G(x,\bar{t}(x),\bar{u}(x))=0$ if and only if $\bar{u}(x)=u(x,\bar{t}(x))$. □

4. Approximate solutions

We keep the notions and assumptions of Section 3. Let $x'=(x_1,\dots,x_{n-1})$.

Proposition 4.1. *Let $\bar{y}(x) = (\bar{y}_1(x),\dots,\bar{y}_N(x))$, $\bar{y}(0)=0$, be a formal solution of the system of analytic equations $f(x,y)=0$, such that the power series $\delta(x,\bar{y}(x))$ is x_n-regular of strictly positive order $p>0$. Then there exists an approximate solution $\bar{v}(x)\in\mathbb{K}[[x']][x_n]^N$ of the system $f(x,y)=0$ and such that $\bar{y}(x)$ is a solution of $f(x,y)=0$ subordinate to $\bar{v}(x)$.*

Proof. By the Weierstrass Preparation Theorem $\delta(x,\bar{y}(x)) = \bar{a}(x)\cdot\text{unit}$ where

$$\bar{a}(x) = x_n^p + \sum_{j=1}^{p}\bar{a}_j(x')x_n^{p-j}$$

is a distinguished polynomial. Using the Weierstrass Division Theorem we get

$$\bar{y}_\nu(x) = \sum_{j=0}^{2p-1}\bar{v}_{\nu,j}(x')x_n^j + \bar{a}(x)^2(c_\nu+\bar{t}_\nu(x)) \text{ for } \nu=1,\dots,N-m$$

and

$$\bar{y}_\nu(x) = \sum_{j=0}^{p-1}\bar{v}_{\nu,j}(x')x_n^j + \bar{a}(x)(c_\nu+\bar{u}_\nu(x)) \text{ for } \nu=N-m+1,\dots,N$$

where $c_\nu\in\mathbb{K}$ for $\nu=1,\dots,N$, while

$$\bar{t}(x) = (\bar{t}_1(x),\dots,\bar{t}_{N-m}(x)) \text{ and } \bar{u}(x) = (\bar{u}_{N-m+1}(x),\dots,\bar{u}_N(x))$$

are formal power series without constant term. Let

$$\bar{v}_\nu(x) = \sum_{j=0}^{2p-1} \bar{v}_{\nu,j}(x')x_n^j + \bar{a}(x)^2 c_\nu \text{ for } \nu = 1, \ldots, N-m$$

and

$$\bar{v}_\nu(x) = \sum_{j=0}^{p-1} \bar{v}_{\nu,j}(x')x_n^j + \bar{a}(x) c_\nu \text{ for } \nu = N-m+1, \ldots, N\,.$$

Clearly $\bar{v}(x) = (\bar{v}_1(x), \ldots, \bar{v}_N(x)) \in \mathbb{K}[[x']][x_n]^N$.

Property 1. $\delta(x, \bar{v}(x)) = \bar{a}(x) \cdot$ *unit*

Proof. From $\bar{y}(x) \equiv \bar{v}(x) \pmod{\bar{a}(x)\mathbf{m_x}}$ we get $\delta(x, \bar{y}(x)) \equiv \delta(x, \bar{v}(x)) \pmod{\bar{a}(x)\mathbf{m_x}}$ and Property 1 follows since $\delta(x, \bar{y}(x)) = \bar{a}(x) \cdot \text{unit}$.

Property 2. $g_i(x, \bar{v}(x)) \equiv 0 \pmod{\bar{a}(x)^2\mathbf{m_x}}$ *for* $i = 1, \ldots, m$.

Proof. Substituting in Taylor's formula (T) $v = \bar{v}(x)$, $h_\nu = \bar{a}(x)^2\bar{t}_\nu(x)$ for $\nu = 1, \ldots, N-m$ and $h_\nu = \bar{a}(x)\bar{u}_\nu(x)$ for $\nu = N-m+1, \ldots, N$ we get

$$\begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} f_1(x, \bar{v}(x)) \\ \vdots \\ f_m(x, \bar{v}(x)) \end{bmatrix} + \bar{a}(x)^2 \frac{J(f_1, \ldots, f_m)}{J(y_1, \ldots, y_{N-m})}(x, \bar{v}(x)) \begin{bmatrix} \bar{t}_1(x) \\ \vdots \\ \bar{t}_{N-m}(x) \end{bmatrix}$$
$$+ \bar{a}(x) J(x, \bar{v}(x)) \begin{bmatrix} \bar{u}_{N-m+1}(x) \\ \vdots \\ \bar{u}_N(x) \end{bmatrix} + \bar{a}(x)^2 \begin{bmatrix} \bar{Q}_1(x) \\ \vdots \\ \bar{Q}_m(x) \end{bmatrix}\,.$$

Multiplying the above identity by $M(x, \bar{v}(x))$ and taking into account the formula

$$M(x, \bar{v}(x))J(x, \bar{v}(x)) = \delta(x, \bar{v}(x))I_m$$

we get Property 2.

From Properties 1 and 2 it follows that

$$g_i(x, \bar{v}(x)) \equiv 0 \pmod{\delta(x, \bar{v}(x))^2\mathbf{m_x}} \qquad \text{for } i = 1, \ldots, m$$

i.e. $\bar{v}(x) \in \mathbb{K}[[x']][x_n]$ is an approximate solution of the system $f(x, y) = 0$. Since

$$\bar{y}_\nu(x) \equiv \bar{v}_\nu(x) \bmod \delta(x, \bar{v}(x))^2\mathbf{m_x} \qquad \text{for } i = 1, \ldots, N-m$$

and

$$\bar{y}_\nu(x) \equiv \bar{v}_\nu(x) \bmod \delta(x, \bar{v}(x))\mathbf{m_x} \qquad \text{for } i = N-m+1, \ldots, N$$

$\bar{y}(x)$ is a subordinate solution to the approximate solution $\bar{v}(x)$. $\square$

Proposition 4.2. *Let $(c^0_{\nu,j})$, $\nu = 1,\dots,N$, $j = 0,1,\dots,D$, be a family of constants such that $c^0_{\nu,0} = 0$ for $\nu = 1,\dots,N$. Suppose that*

$$\left(\sum_{j=0}^{D} c^0_{1,j} x_n^j, \dots, \sum_{j=0}^{D} c^0_{N,j} x_n^j\right)$$

is an approximate solution of the system of equations $f(0,x_n,y) = 0$ such that

$$\operatorname{ord}\delta\left(0, x_n, \sum_{j=0}^{D} c^0_{1,j} x_n^j, \dots, \sum_{j=0}^{D} c^0_{N,j} x_n^j\right) = p\,, \quad 0 < p < +\infty\,.$$

Let $V^0 = (V^0_{\nu,j})$, $\nu = 1,\dots,N$, $j = 0,1,\dots,D$ be variables. Then there exists a sequence

$$F(x', V^0) = (F_1(x', V^0), \dots, F_M(x', V^0)) \in \mathbb{K}\{x', V^0\}^M$$

such that for any family $(\bar v^0_{\nu,j}(x'))$ of formal power series without constant term the following two conditions are equivalent

(i) $\left(\sum_{j=0}^{D}(c^0_{1,j} + \bar v^0_{1,j}(x'))x_n^j, \dots, \sum_{j=0}^{D}(c^0_{N,j} + \bar v^0_{N,j}(x'))x_n^j\right)$ *is an approximate solution of the system $f(x,y) = 0$,*

(ii) $F(x', (\bar v^0_{\nu,j}(x'))) = 0$ *in $\mathbb{K}[[x']]$.*

Proof. Let

$$v_{\bar\nu}(x_n) = \sum_{j=0}^{D}(c^0_{\nu,j} + V^0_{\nu,j})x_n^j, \quad v(x_n) = (v_1(x_n), \dots, v_N(x_n))\,.$$

It is easy to check that $\delta(x, v(x_n))$ is x_n-regular of order p. By the Weierstrass Division Theorem

$$g_i(x, v(x_n)) = Q_i(x, V^0)\delta(x, v(x_n))^2 + \sum_{j=0}^{2p-1} R_{i,j}(x', V^0)x_n^j$$

for $i = 1,\dots,m$. Let

$$\bar v_\nu(x) = \sum_{j=0}^{D}(c^0_{\nu,j} + \bar v^0_{\nu,j})x_n^j, \quad \bar v(x) = (\bar v_1(x), \dots, \bar v_N(x))$$

where $(\bar v^0_{\nu,j}(x'))$ is a family of formal power series without constant term. Thus we get

$$g_i(x, \bar v(x)) = Q_i(x, \bar v(x))\delta(x, \bar v(x))^2 + \sum_{j=0}^{2p-1} R_{i,j}(x', \bar v^0_{\nu,j}(x'))x_n^j \text{ for } i = 1,\dots,m\,.$$

By the uniqueness of the remainder in the Weierstrass Division Theorem we have that $\bar v(x)$ is an approximate solution of the system of analytic equations $f(x,y) = 0$

if and only if $R_{i,j}(x',(\bar{v}^0_{\nu,j}(x'))) = 0$ for $i = 1,\ldots,m$ and $j = 0,1,\ldots,2p-1$ in $\mathbb{K}[[x']]$. This proves the proposition. $\square$

5. Proof of the theorem (by induction on the number n of variables x)

The theorem is trivial for $n = 0$. Suppose that $n > 0$ and that the theorem is true for $n-1$. By Proposition 2.1 we may suppose that $\bar{y}(x)$ is a simple solution of the system $f(x,y) = 0$. Let

$$\delta(x,y) = \det \frac{J(f_1,\ldots,f_m)}{J(y_{N-m+1},\ldots,y_N)} \,.$$

Without diminishing the generality we may suppose that $\delta(x,\bar{y}(x)) \neq 0$. If $\delta(0,0) \neq 0$ then the theorem follows from the Implicit Function Theorem. Suppose that $\delta(0,0) = 0$. After a linear change of the variables $x_1,\ldots,x_n$ we may assume that $\delta(x,\bar{y}(x))$ is x_n-regular of order $p > 0$. By Proposition 4.1 the system of equations $f(x,y) = 0$ has an approximate solution $\bar{v}(x) = (\bar{v}_1(x),\ldots,\bar{v}_N(x)) \in \mathbb{K}[[x']][x_n]^N$ such that the solution $\bar{y}(x)$ is subordinate to $\bar{v}(x)$. Write

$$\bar{v}_\nu(x) = \sum_{j=0}^{D}(c^0_{\nu,j} + \bar{v}^0_{\nu,j}(x'))x_n^j, \quad D \geq 0 \text{ an integer}$$

where $(\bar{v}_{\nu,j}(x'))$ is a family of formal power series without constant term. It is easy to check that

$$\left(\sum_{j=0}^{D} c^0_{1,j}x_n^j,\ldots,\sum_{j=0}^{D} c^0_{N,j}x_n^j\right)$$

is an approximate solution of the system $f(0,x_n,y) = 0$ such that

$$\operatorname{ord}\delta\left(0,x_n,\sum_{j=0}^{D} c^0_{1,j}x_n^j,\ldots,\sum_{j=0}^{D} c^0_{N,j}x_n^j\right) = p\,.$$

By Proposition 4.2 there exist convergent power series $F(x',V^0) \in \mathbb{K}\{x',V^0\}^M$ such that $F(x',(\bar{v}^0_{\nu,j}(x'))) = 0$. By induction hypothesis there exist convergent power series $(V^0_{\nu,j}(x',s))$ in $\mathbb{K}\{x',s\}$, where $s = (s_1,\ldots,s_q)$ are new variables and formal power series $\bar{s}(x') = (\bar{s}_1(x'),\ldots,\bar{s}_q(x'))$ without constant term such that

$$F(x',(V^0_{\nu,j}(x',s))) = 0, \quad V^0_{\nu,j}(x',\bar{s}(x')) = \bar{v}^0_{\nu,j}(x')\,.$$

Let

$$v_\nu(x,s) = \sum_{j=0}^{D}(c^0_{\nu,j} + V^0_{\nu,j}(x',s))x_n^j \text{ for } \nu = 1,\ldots,N$$

and $v(x,s) = (v_1(x,s),\ldots,v_N(x,s))$. Thus $\bar{v}_\nu(x) = v_\nu(x,\bar{s}(x'))$ for $\nu = 1,\ldots,N$. Again by Proposition 4.2 $v(x,s)$ is an approximate solution of the system

$f(x,y)=0$. By the Bourbaki-Tougeron Implicit Function Theorem the system $f(x,y)=0$ has the parametric solution determined by $v(x,s)$:

$$y_\nu(x,s,t) = v_\nu(x,s) + \delta(x,v(x,s))^2 t_\nu \text{ for } \nu = 1,\dots,N-m$$
$$y_\nu(x,s,t) = v_\nu(x,s) + \delta(x,v(x,s)) u_\nu(x,s,t) \text{ for } \nu = N-m+1,\dots,N\ .$$

On the other hand

$$\bar y_\nu(x,t) = \bar v_\nu(x) + \delta(x,\bar v(x))^2 t_\nu \text{ for } \nu = 1,\dots,N-m$$
$$\bar y_\nu(x,t) = \bar v_\nu(x) + \delta(x,\bar v(x)) \bar u_\nu(x,t) \text{ for } \nu = N-m+1,\dots,N$$

is the parametric solution determined by $\bar v(x)$. Since the formal solution $\bar y(x)$ is subordinate to the approximate solution $\bar v(x)$ there exist formal power series $\bar t(x) = (\bar t_1(x),\dots,\bar t_{N-m}(x))$, $\bar t(0)=0$, such that $\bar y(x) = \bar y(x,\bar t(x))$. We have

$$y_\nu(x,,\bar s(x'),t) = \bar v_\nu(x) + \delta(x,\bar v(x))^2 t_\nu \text{ for } \nu = 1,\dots,N-m$$
$$y_\nu(x,\bar s(x'),t) = \bar v_\nu(x) + \delta(x,\bar v(x)) u_\nu(x,\bar s(x'),t) \text{ for } \nu = N-m+1,\dots,N$$

By the uniqueness of the parametric solution determined by the approximate solution $\bar v(x)$ we get

$$y(x,\bar s(x'),\bar t(x)) = \bar y(x,\bar t(x)) = \bar y(x).$$

□

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