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BÉZOUT'S INEQUALITY FOR REAL POLYNOMIALS

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Abstract. Let $F(X,Y)$, $G(X,Y)$ be polynomials of degrees $m, n > 0$ respectively. We prove, that the set $\{(x,y) \in \mathbb{R}^2 : F(x,y) = G(x,y) = 0\}$ has at most mn connected components.

Classical Bézout's theorem says that the number of complex solutions of a system of n complex polynomial equations with n variables does not exceed the product of degrees of the polynomials, provided it is finite. An elementary proof of the theorem for $n = 2$ can be found in [3], chapter X, §3.2. For real polynomials such a bound doesn't hold; here is the example given by Fulton [1]: the system of equations

$$\prod_{i=1}^{m}(x-i)^2 + \prod_{j=1}^{m}(y-j)^2 = 0, \quad xz = 0, \quad yz = 0$$

has m^2 solutions in $\mathbb{R}^3$, while the product of equations degrees is equal to $2m \cdot 2 \cdot 2 = 8m < m^2$ for $m > 8$.

Our aim is to show that there is no such an example in the case of two polynomial equations with two unknowns.

We will prove the following

Theorem. *If polynomials $F(X,Y)$, $G(X,Y) \in \mathbb{R}[X,Y]$ have degrees $m, n > 0$ respectively, then the set of solutions of a system of equations $F(X,Y) = G(X,Y) = 0$ has at most mn connected components.*

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The proof is based on two lemmas

Lemma 1. *If polynomials F, G with degrees $m, n > 0$ respectively are coprime then the system $F(X,Y) = G(X,Y) = 0$ has at most $m \cdot n$ real solutions.*

Proof: If F, G are coprime in $\mathbb{R}[X,Y]$ then they are also coprime in $\mathbb{C}[X,Y]$; hence the system $F = G = 0$ has at most mn solutions in $\mathbb{C}^2$ and in particular in $\mathbb{R}^2$. $\square$

Lemma 2. *If $P \in \mathbb{R}[X,Y]$ is not constant then the set $\{(x,y) : P(x,y) = 0\}$ has at most $(\deg P)^2$ connected components.*

Proof. It is sufficient to prove Lemma 2 for P irreducible.

In fact: suppose that $P = P_1 \cdot \dots \cdot P_s$ where P_i, $i = 1, \dots, s$ are irreducible and assume that Lemma 2 is true for every P_i. Then the number of components of the set $P = 0$ does not exceed the sum of the numbers of component of the sets $P_i = 0$. Hence the number of connected components of the set $P = 0$ is less than or equal to $\sum_{i=1}^{s} (\deg P_i)^2 \leqslant \left(\sum_{i=0}^{s} \deg P_i\right)^2 = (\deg P)^2$.

Suppose that P is an irreducible polynomial of positive degree and take a point (a,b) such that $P(a,b) \neq 0$. Put $Q(X,Y) = (X-a)^2 + (Y-b)^2$ and consider the Jacobian determinant $J(P,Q)$ of polynomials P, Q.

If $J(P,Q) = 2(Y-b)P_X - 2(X-b)Q_Y \neq 0$ in $\mathbb{R}[X,Y]$ then the polynomials P, $J(P,Q)$ are coprime otherwise we would have $J(P,Q) = \mathit{const}\, P$ since P is prime, which is impossible because $P(a,b) \neq 0$ and $J(P,Q)(a,b) = 0$.

We will show that any connected component M of the set $P = 0$ intersects the curve $J(P,Q) = 0$. Let (x_0, y_0) be a point of M in which the polynomial Q reaches its minimum on M. If $(x_0, y_0) \in M$ is a critical point of P then of course $J(P,Q)(x_0, y_0) = 0$. If it is not a critical point then $J(P,Q)(x_0,y_0) = 0$ by the method of Lagrange multipliers, [2], p. 152. Hence the number c of connected components of the set $P = 0$ is not greater than the number of solutions of the system $P = J(P,Q) = 0$. We have $c \leqslant (\deg P)^2$ by Lemma 1.

Let us consider the case $J(P,Q) = 0$ in $\mathbb{R}[X,Y]$. We have

Property. *If $P \in \mathbb{R}[X,Y]$ and $J(P,Q) = 0$ in $\mathbb{R}[X,Y]$ then $P(X,Y) = P_0(Q(X,Y))$ for some $P_0(T) \in \mathbb{R}[T]$.*

Proof of the property. Put $U = X - a$, $V = Y - b$. Then the assumption of the property can be rewritten in the form $J(P,Q) = 2(VP_U - UP_V) = 0$ in $\mathbb{R}[U,V]$. Let us put $DF = VF_U - UF_V$ for any polynomial $F \in \mathbb{R}[U,V]$.

We have

1) if $F(U,V)$ is a homogeneous polynomial of degree $n > 0$ than DF also,
2) if $F = (U^2+V^2)^k \tilde{F}$ then $DF = (U^2+V^2)^k D\tilde{F}$,
3) if $F \neq \mathit{const}$ is a homogeneous form and $DF = 0$ then U^2+V^2 divides F.

To check 3) let us note that the conditions $VF_U - UF_V = 0$ and $UF_U + VF_V = (\deg F)F$ imply the equality $(U^2+V^2)F_U = (\deg F)UF$. Since polynomials U^2+V^2, $(\deg F)U$ are coprime we have that U^2+V^2 divides F.

Now let $P \in \mathbb{R}[U,V]$ be such that $DP = 0$. If $P = \sum P_j$ with P_j homogeneous of degree j then $DP_j = 0$. The conditions 2) and 3) give that $P_j = c_j(U^2+V^2)^{\frac{j}{2}}$ for j even and $P_j = 0$ for j odd.

To complete the proof of Lemma 2 in the case $J(P,Q) = 0$ in $\mathbb{R}[X,Y]$ note that by Property we have $P = P_0(Q)$, where P_0 is a polynomial of one variable. Therefore the set $P = 0$ consists of a finite number of circles. The number of circles does not exceed $\deg P_0 < \deg P$. $\square$

Proof of Theorem. If F, G are coprime then Theorem is true by Lemma 1. Suppose that $P = \mathrm{GCD}(F,G)$ is of positive degree. Then

$$\{F = G = 0\} = \{\frac{F}{P} = \frac{G}{P} = 0\} \cup \{P = 0\}.$$

Put $k = \deg P$. By Lemma 1 the set $\{\frac{F}{P} = \frac{G}{P} = 0\}$ has at most $(m-k)(n-k)$ connected components. Hence the set under consideration has at most

$$(m-k)(n-k) + k^2 = mn - k(m-k+n-k) \leqslant mn$$

connected components. $\square$

References

[1] W. Fulton, *Intersection Theory*, Springer, Berlin 1984.

[2] F. Leja, *Rachunek różniczkowy i całkowy*, Tenth edition, PWN, Warszawa 1969 (in Polish).

[3] A. Mostowski and M. Stark, *Elementy algebry wyższej*, Ninth edition, PWN, Warszawa 1974 (in Polish).

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