Revista de la Real Academia Canaria de Ciencias
Volume XXX, 2018 pp. 31-54

POLYNOMIAL AUTOMORPHISMS OF THE AFFINE PLANE AND SINGULARITIES OF ALGEBRAIC CURVES

Evelia R. García Barroso

Departamento de Matemáticas, Estadística e I.O.
Sección de Matemáticas, Universidad de La Laguna
Apartado de Correos 456
38200 La Laguna, Tenerife, España

Arkadiusz Płoski

Department of Mathematics and Physics
Kielce University of Technology
Al. 1000 L PP7
25-314 Kielce, Poland

Abstract. Abhyankar and Moh achieved a major breakthrough in the global geometry of the affine plane with their papers [2] and [3]. The aim of this expository article is to provide an introduction to the Abhyankar-Moh theory. We base our approach on the local theory of algebraic plane curves explained in our previous paper [18], where we reproved the basic properties of approximate roots without resorting to Puiseux series. We pass then to the projective closure of the affine plane in order to prove the Embedding Line Theorem [3] and related results such as the Moh-Ephraim Pencil Theorem and the Abhyankar-Moh Semigroup Theorem.

1. **Introduction.** The aim of this expository article is to present some applications of the local theory of plane algebraic curves to the global geometry of the affine plane over algebraically closed field K of arbitrary characteristic. Abhyankar and Moh in their fundamental papers [2] and [3] studied the semigroup of a meromorphic curve using the Newton-Puiseux expansions and the concept of approximate roots of polynomials in order to prove the famous *Embedding Line Theorem.* Following their ideas we provide an introduction to the main theorem of [3]. Our proof uses basic results of the theory of branches of plane algebraic curves explained in [18].

The contents of this article are

1. Preliminaries
2. Affine curves isomorphic to an affine line

2010 *Mathematics Subject Classification.* 4H20; 14R10; 14H37.

Key words and phrases. Polynomial automorphism, curve with one branch at infinity, embedding line Theorem, Abhyankar-Moh theory, approximate root.

The first-named author was partially supported by the Spanish Project MTM 2016-80659-P.

3. Polynomial automorphisms of the affine plane
4. Numerical semigroups and plane branches
5. Abhyankar-Moh theory of approximate roots and the Embedding Line Theorem
6. Curves with one branch at infinity and the Abhyankar-Moh Semigroup Theorem

2. **Preliminaries.** In this section we fix our notations and recall basic notions and results of the theory of plane algebraic curves. The references to this section are [21, chapters 1-4] and [24, sections 1-6].

2.1. **Affine plane curves.** Let $f = f(x,y) = \sum_{\alpha,\beta} c_{\alpha,\beta} x^\alpha y^\beta \in K[x,y]$ be a polynomial with coefficients in an algebraically closed field K. We put

- $\operatorname{supp} f = \{(\alpha,\beta) \in \mathbf{N}^2 \ : \ c_{\alpha,\beta} \neq 0\}$,
- $\deg f = \sup\{\alpha+\beta \ : \ (\alpha,\beta) \in \operatorname{supp} f\}$,
- $f^+ = \sum_{\alpha+\beta=\deg f} c_{\alpha,\beta} x^\alpha y^\beta$.

By conventions $\deg 0 = -\infty$ and $0^+ = 0$. We have $\deg fg = \deg f + \deg g$ and $(fg)^+ = f^+ g^+$ for any $f, g \in K[x,y]$. If $P = (a,b) \in K^2$ then we put $f(P) = f(a,b)$. Since the field K is infinite we may identify the polynomial f and the function $P \to f(P)$. Put $V(f) = \{P \in K^2 \ : \ f(P) = 0\}$. A set $\Gamma \subset K^2$ is an *affine (plane) curve* if there is a nonconstant polynomial f such that $\Gamma = V(f)$. If f is of minimal degree then we call f as *minimal polynomial* of Γ. It is uniquely determined by Γ up to a constant factor. We put $\deg \Gamma = \deg f$ (the *degree* of Γ). An affine line is an affine curve of degree one. An affine curve is *irreducible* if its minimal polynomial is irreducible in $K[x,y]$.

A point $P \in \Gamma$ is a *singular* point of Γ if $\frac{\partial f}{\partial x}(P) = 0$ and $\frac{\partial f}{\partial y}(P) = 0$ for the minimal polynomial f of Γ. Otherwise it is called *simple* or *nonsingular*. The set of singular points of an affine curve is finite [24, Corollary 6.9]. If that set is empty, the curve is called *nonsingular*.

2.2. **Algebroid curves.** We use formal power series to study the local properties of algebraic curves. The reader will find the proofs omitted in this section in [9] and [28]. Recall that the ring of formal power series $K[[x,y]]$ is a unique factorization domain.

Let $f = f(x,y) = \sum_{\alpha,\beta} c_{\alpha,\beta} x^\alpha y^\beta \in K[[x,y]]$ be a formal power series with coefficients in K. We put

- $\operatorname{supp} f = \{(\alpha,\beta) \in \mathbf{N}^2 \ : \ c_{\alpha,\beta} \neq 0\}$,
- $\operatorname{ord}\, f = \inf\{\alpha+\beta \ : \ (\alpha,\beta) \in \operatorname{supp} f\}$,
- $\operatorname{in} f = \sum_{\alpha+\beta=\operatorname{ord}\, f} c_{\alpha,\beta} x^\alpha y^\beta$.

Observe that $f(0,0) = c_{0,0}$ (the constant term of f). By conventions $\operatorname{ord}\, 0 = +\infty$ and $\operatorname{in} 0 = 0$. We have $\operatorname{ord}\, fg = \operatorname{ord}\, f + \operatorname{ord}\, g$, $\operatorname{in} fg = \operatorname{in} f \operatorname{in} g$ and $(fg)(0,0) = f(0,0)g(0,0)$, for any $f, g \in K[[x,y]]$.

A power series $u \in K[[x,y]]$ is a *unit* if $uv = 1$ for a power series $v \in K[[x,y]]$. Note that u is a unit if and only if $u(0,0) \neq 0$ (that is $\operatorname{ord}\, u = 0$).

Let $f \in K[[x,y]]$ be a nonzero power series without constant term. An *algebroid curve* $\{f=0\}$ is by definition the ideal $(f)K[[x,y]]$ generated by f in $K[[x,y]]$. We say that $\{f=0\}$ is *irreducible* (*reduced*) if f is irreducible (has no multiple factors) in $K[[x,y]]$. The irreducible algebroid curves are also called *branches*.

A *formal isomorphism* Φ is a pair of power series $\Phi(x,y) = (ax+by+\cdots, a'x+b'y+\cdots) \in K[[x,y]]^2$, where $ab'-a'b \neq 0$ and the dots mean terms in x and y of order greater than 1. The map $f \to f\circ\Phi$ is an isomorphism of the ring $K[[x,y]]$. Two algebroid curves $\{f=0\}$ and $\{g=0\}$ are said to be *formally equivalent* if there is a formal isomorphism Φ such that $f\circ\Phi = g\cdot\mathrm{unit}$.

Let t be a variable. A *parametrization* is a pair $(\varphi(t),\psi(t))$ of formal power series in $K[[t]]$ such that $\varphi(0)=\psi(0)=0$ and $\varphi(t)\neq 0$ or $\psi(t)\neq 0$. Two parametrizations $(\varphi(t),\psi(t))$ and $(\varphi_1(t_1),\psi_1(t_1))$ are *equivalent* if there exists a formal power series $\tau(t)\in K[[t]]$ with $\operatorname{ord}\tau(t)=1$ such that $\varphi(t)=\varphi_1(\tau(t))$, $\psi(t)=\psi_1(\tau(t))$. A *parametrization* $(\varphi(t),\psi(t))\in K[[t]]^2$ is *good* if there does not exist $\tau(t)\in K[[t]]$, $\operatorname{ord}\tau(t)>1$ and a parametrization $(\varphi_1(t_1),\psi_1(t_1))$ such that $\varphi(t)=\varphi_1(\tau(t))$ and $\psi(t)=\psi_1(\tau(t))$.

Theorem 2.1. *(The Normalization Theorem) Let $f=f(x,y)\in K[[x,y]]$ be an irreducible power series. Then there is a good parametrization $(\varphi(t),\psi(t))$ such that $f(\varphi(t),\psi(t))=0$. Any two such parametrizations are equivalent.*

Remark 2.2. With the notations introduced above we get $\operatorname{ord} f(x,0)=\operatorname{ord}\psi(t)$ and $\operatorname{ord} f(0,y)=\operatorname{ord}\varphi(t)$. If $(\varphi(t),\psi(t))$ is a good parametrization and a power series $f=f(x,y)\in K[[x,y]]$ satisfies the conditions $f(\varphi(t),\psi(t))=0$, $\operatorname{ord} f(x,0)=\operatorname{ord}\psi(t)$ and $\operatorname{ord} f(0,y)=\operatorname{ord}\varphi(t)$ then $f=f(x,y)$ is irreducible. $\diamond$

Let $f,g\in K[[x,y]]$ be nonzero power series without constant term. Let $f=f_1\cdots f_m$ in $K[[x,y]]$ with irreducible factors f_i, $i=1,\dots,m$. Let $(\varphi_i(t_i),\psi_i(t_i))$ be a good parametrization such that $f_i(\varphi_i(t_i),\psi_i(t_i))=0$ in $K[[t_i]]$. Then, we define $i_0(f,g)=\sum_{i=1}^m \operatorname{ord} g(\varphi_i(t_i),\psi_i(t_i))$ the *intersection multiplicity* or *intersection number* of the algebroid curves $\{f=0\}$ and $\{g=0\}$. If $f(0,0)\neq 0$ or $g(0,0)\neq 0$ we put $i_0(f,g)=0$. The following properties of intersection multiplicity are basic:

(i) $i_0(f,g)=+\infty$ if and only if f and g have a common factor in $K[[x,y]]$,
(ii) $i_0(f,g_1g_2)=i_0(f,g_1)+i_0(f,g_2)$,
(iii) $i_0(f,g+hf)=i_0(f,g)$,
(iv) $i_0(f,g)=i_0(g,f)$,
(v) if Φ is a formal isomorphism then $i_0(f\circ\Phi,g\circ\Phi)=i_0(f,g)$,
(vi) $i_0(f,g)=1$ if and only if $\operatorname{jac}(f,g)(0,0)=\frac{\partial f}{\partial x}(0,0)\frac{\partial g}{\partial y}(0,0)-\frac{\partial f}{\partial y}(0,0)\frac{\partial g}{\partial x}(0,0)\neq 0$.

The following property of the intersection multiplicity is the main ingredient of the proof of the Jung-van der Kulk theorem (see Proposition 4.7 in Section 4 of this paper).

(vii) Let $f,g,l\in K[[x,y]]$ be irreducible power series, where $\operatorname{ord} l=1$. Let $m=i_0(f,l)$, $n=i_0(g,l)$ and $d=\gcd(m,n)$. Then

$$i_0(f,g)\equiv 0 \pmod{\tfrac{m}{d} \text{ or } \tfrac{n}{d}}.$$

For the proof see [19].

2.3. **Projective plane curves.** Let $\mathbf{P}^2(K)$ be the projective plane over K. For any homogeneous polynomial $H = H(X,Y,Z) \in K[X,Y,Z]$ of positive degree we put $V(H) = \{(a:b:c) \in \mathbf{P}^2(K) \;:\; H(a,b,c) = 0\}$ and any set of the form $V(H)$ a *projective (plane) curve*. Let C be a projective curve. Then, a *minimal polynomial* of C is, by definition, a homogeneous polynomial H of minimal degree such that $C = V(H)$. We put $\deg C = \deg H$. A *projective line* is a projective curve of degree one. A projective curve is irreducible if its minimal polynomial H is irreducible in $K[X,Y,Z]$. Fix a line *at infinity* $L_\infty = V(Z)$. We identify the affine plane K^2 and the set $\mathbf{P}^2(K)\backslash L_\infty$ by introducing the affine coordinates $x = \frac{X}{Z}$, $y = \frac{Y}{Z}$. The points of $\mathbf{P}^2(K)\backslash L_\infty$ are called *points at finite distance*. For any nonconstant polynomial $f(x,y) \in K[x,y]$ we define the homogeneous form $F(X,Y,Z) = Z^{\deg f} f(\frac{X}{Z}, \frac{Y}{Z})$. The *projective closure* of $V(f)$ is equal to $V(F)$. If f is a minimal polynomial of $V(f)$, then F is a minimal polynomial of $V(F)$. In particular $\deg V(f) = \deg V(F)$. The points at infinity of $V(F)$ satisfy the equations $f^+(x,y) = 0, z = 0$. To study properties of a projective curve C near $P = (a:b:0) \in L_\infty$ we use the affine coordinates $(u,v) = \left(\frac{Y}{X}, \frac{Z}{X}\right)$ if $a \neq 0$ or $(s,t) = \left(\frac{X}{Y}, \frac{Z}{Y}\right)$ if $b \neq 0$.

Let C, D be projective curves intersecting in a finite number of points. Let $P \in C \cap D$ and consider a system of affine coordinates (ξ, η) centered at P, that is such that $(\xi(P), \eta(P)) = (0,0)$. Let $f(\xi,\eta)$ (respectively $g(\xi,\eta)$) be the minimal polynomial of C (respectively of D) in the coordinates (ξ,η). Then the *intersection multiplicity* $i(C,D;P)$ of C and D at P is defined to be the intersection multiplicity $i_0(f,g)$ of the algebroid curves $\{f=0\}$ and $\{g=0\}$. It is independent on the choice of affine coordinates.

Theorem 2.3. *(Bézout's theorem) With the above notations we get*

$$\sum_{P \in C \cap D} i(C,D;P) = \deg C \cdot \deg D.$$

For the proof we refer the reader to [34].

For any point P of a projective curve C we define the *multiplicity of C at P* as

$$\mathrm{mult}_P(C) = \inf\{i(C,L;P) \;:\; L \not\subset C \text{ is a line passing through } P\}.$$

A line L passing through P is *tangent to C at P* if $i(C,L;P) > \mathrm{mult}_P(C)$. Let (ξ,η) be affine coordinates centered at P and let $f(\xi,\eta) = 0$ be the affine equation of C. Then $\mathrm{mult}_P(C) = \mathrm{ord}\, f$ and the tangents to C at P have the equation $\mathrm{in} f = 0$ in the coordinates ξ, η.

By a *curve with multiple components* we mean a formal linear combination $C = m_1C_1 + \cdots + m_kC_k$, where the C_i are irreducible curves and the m_i are natural numbers. If $F_i = 0$ is a minimal polynomial of C_i, then the *minimal equation* of C is, by definition, $F = F_1^{m_1} \cdots F_k^{m_k} = 0$. In the sequel we identify the curves with multiple components and homogeneous polynomials (up to a constant factor). The notion of intersection multiplicity and Bézout's theorem extend to the case of curves with multiple components. Instead of $i(C,D;P)$ we also write $i(F,G;P)$, where $F = 0$ (respectively $G = 0$) are the minimal equations of C (respectively of

D). If $f, g \in K[x, y]$ then $i(f, g; P) := i(F, G; P)$, where F, G are the homogenous polynomials corresponding to f and g.

3. **Affine curves isomorphic to an affine line.** A polynomial mapping $(p, q) : K \longrightarrow K^2$ is a *polynomial embedding (of the line K)* if there is a polynomial map $g : K^2 \longrightarrow K$ such that $g(p(t), q(t)) = t$ in $K[t]$. This is equivalent to $K[p(t), q(t)] = K[t]$. An affine curve $\Gamma \subset K^2$ is isomorphic to the line K (such curves will be called *embedded lines*) if there exists a polynomial embedding $(p, q) : K \longrightarrow K^2$ such that $(p, q)(K) = \Gamma$. We call the pair (p, q) a *polynomial parametrization* of Γ. It is easy to check that any embedded line is an irreducible affine curve. The graph of a polynomial of one variable is an embedded line.

To prove basic properties of embedded lines we need several lemmas. Let Γ be an embedded line with a minimal equation $f(x, y) = 0$ and let $(p(t), q(t))$ be a polynomial parametrization of Γ. Suppose that $(p(0), q(0)) = (0, 0)$.

Lemma 3.1. *Let $(p_1(s), q_1(s)) \in K[[s]]^2$ be a parametrization of the algebroid curve $f(x, y) = 0$. Then $(p_1(s), q_1(s)) = (p(\tau(s)), q(\tau(s)))$, where $\tau(s) \in K[[s]]$ and $\tau(0) = 0$.*

Proof. The polynomials $x - p(t)$, $y - q(t)$ vanish on the set of solutions of the system of equations $f(x, y) = 0$, $g(x, y) - t = 0$. Thus by Hilbert's Nullstellensatz $x - p(t)$ and $y - q(t)$ belong to the radical of the ideal generated by $f(x, y)$ and $g(x, y) - t$ in $K[x, y, t]$. Let $\tau(s) := g(p_1(s), q_1(s))$. Then $f(p_1(s), q_1(s)) = 0$ and $g(p_1(s), q_1(s)) - \tau(s) = 0$, which implies $p_1(s) - p(\tau(s)) = 0$ and $q_1(s) - q(\tau(s)) = 0$. □

Lemma 3.2. *The polynomial $f(x, y)$ is irreducible in $K[[x, y]]$.*

Proof. Let $f_0(x, y) \in K[[x, y]]$ be an irreducible power series such that $f_0(p(t), q(t)) = 0$. Then $f(x, y) = f_0(x, y)^k f_1(x, y) \in K[[x, y]]$, where $k \geq 1$ is an integer and $f_1(p(t), q(t)) \neq 0$ in $K[[t]]$. We claim that $f_1(0, 0) \neq 0$. Otherwise, applying the Normalization Theorem to an irreducible factor of f_1, there would exist a parametrization $(p_1(s), q_1(s)) \in K[[s]]^2$ such that $f_1(p_1(s), q_1(s)) = 0$. Thus $f(p_1(s), q_1(s)) = 0$ and $p_1(s) = p(\tau(s))$, $q_1(s) = q(\tau(s))$, with $\tau(s) \in K[[s]]$, $\tau(0) = 0$ by Lemma 3.1. From $f_1(p(\tau(s)), q(\tau(s))) = 0$ we get $f_1(p(t), q(t)) = 0$, which is a contradiction.

It remains to check that $k = 1$. If we had $k > 1$ then the power series f_0^{k-1} would divide $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$, which implies that $\frac{\partial f}{\partial x}(p(t), q(t)) = 0$ and $\frac{\partial f}{\partial y}(p(t), q(t)) = 0$, that is $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ vanish on Γ. This is impossible since an algebraic curve has a finite number of singular points. □

Lemma 3.3. *With the above notations we get* $\operatorname{ord} f(x, y) = 1$.

Proof. Since $g(p(t), q(t)) = t$ we have that $(p(t), q(t))$ is a good parametrization of the algebroid curve $\{f(x, y) = 0\}$. By Lemma 3.2 $f(x, y)$ is irreducible. Therefore $\operatorname{ord} f = \inf\{\operatorname{ord} p(t), \operatorname{ord} q(t)\}$ (see Remark 2.2). From $g(p(t), q(t)) = t$ we get $\frac{\partial g}{\partial x}(p(t), q(t))p'(t) + \frac{\partial g}{\partial y}(p(t), q(t))q'(t) = 1$, whence $(p'(0), q'(0)) \neq (0, 0)$. Thus $\inf\{\operatorname{ord} p(t), \operatorname{ord} q(t)\} = 1$ and consequently $\operatorname{ord} f = 1$ □

Now we can check

Proposition 3.4. *Any embedded line is nonsingular.*

Proof. Let Γ be an embedded line with a polynomial parametrization $(p(t), q(t))$. Let $P = (x_0, y_0) \in \Gamma$ and $(x_0, y_0) = (p(t_0), q(t_0))$ for a $t_0 \in K$.

Introducing the new coordinates $\tilde{x} = x - x_0$ and $\tilde{y} = y - y_0$ and replacing the parameter t by $\tilde{t} = t - t_0$ we may assume that $P = (0, 0)$ and $(p(0), q(0)) = (0, 0)$. By Lemma 3.3 we get $\operatorname{ord} f(x, y) = 1$, which is equivalent to $\frac{\partial f}{\partial x}(0, 0) \neq 0$ or $\frac{\partial f}{\partial y}(0, 0) \neq 0$. □

Proposition 3.5. *Let Γ be an embedded line of degree greater than 1 and let $(p(t), q(t))$ be a polynomial parametrization of Γ. Let L be the line $bx - ay + c = 0$. Then*

$$i(\Gamma, L; (p(t_0), q(t_0))) = \operatorname{ord}\,(bp(t + t_0) - aq(t + t_0) + c).$$

In particular, the number of points of the intersection of Γ and L counted with their multiplicities is equal to $\deg(bp(t) - aq(t) + c)$.

Proof. Let $f(x, y) = 0$ be the minimal equation of Γ. Like in the proof of Proposition 3.4 we may assume that $t_0 = 0$ and $(p(0), q(0)) = (0, 0)$. Then L is the line $bx - ay = 0$. Since by Lemma 3.2 the algebroid curve $\{f(x, y) = 0\}$ is irreducible and $(p(t), q(t))$ is a good parametrization, so we get $i(\Gamma, L; (0, 0)) = \operatorname{ord}\,(bp(t) - aq(t))$. □

Proposition 3.6. *Let Γ be an embedded line of degree greater than one and let $(p(t), q(t)) = (a_0t^n + \cdots, b_0t^n + \cdots)$ with $(a_0, b_0) \neq (0, 0)$ be a polynomial parametrization of Γ. Then*

1. $\deg \Gamma = \max\{\deg p, \deg q\} = n$,
2. *the closure $\overline{\Gamma}$ in $\mathbf{P}^2(K)$ intersects the line at infinity exactly at the point $O_\infty = (a_0 : b_0 : 0)$,*
3. $\operatorname{mult}_{O_\infty} \overline{\Gamma} = n - \deg(b_0p(t) - a_0q(t))$ *and the line at infinity L_∞ is the only tangent to $\overline{\Gamma}$ at O_∞.*

Proof. Let L be the line $V(bx - ay + c)$. Then the projective closure $\overline{L}$ of L intersects the line at infinity in the point $(a : b : 0)$. If $(a : b : 0) \neq O_\infty = (a_0 : b_0 : 0)$ then, by Proposition 3.5, the number of points of the intersection of $\overline{\Gamma}$ and $\overline{L}$ counted with their multiplicities is equal to $\deg(bp(t) - aq(t)) = n$ and by Bézout's theorem $\deg \Gamma = \deg \overline{\Gamma} = n$. Moreover $\overline{\Gamma} \cap L_\infty = \{O_\infty\}$. If $\overline{L}$ passes through

O_∞ then $L = V(b_0x - a_0y + c_0)$, where $c_0 \in K$ and again by Bézout's theorem $i(\overline{\Gamma}, \overline{L}; O_\infty) = n - \deg(b_0p(t) - a_0q(t)) < n$ for all lines $\overline{L} \neq L_\infty$ passing through O_∞. Since $\overline{\Gamma} \cap L_\infty = \{O_\infty\}$ we have that $i(\overline{\Gamma}, L_\infty; O_\infty) = n$. Therefore $\mathrm{mult}_{O_\infty}\overline{\Gamma} = n - \deg(b_0p(t) - a_0q(t))$ and the line at infinity L_∞ is the unique tangent to $\overline{\Gamma}$ at O_∞. □

Now, let $C \subset \mathbf{P}^2(K)$ be a projective curve. We say that C is *analytically irreducible* at $P \in C$ if the affine equation of C in a system of affine coordinates (x, y) centered at P is irreducible in $K[[x, y]]$. Clearly a projective curve C is analytically irreducible at any nonsingular point of C.

Proposition 3.7. *Let Γ be an embedded line of degree greater than one and let O_∞ be the only point at infinity of $\overline{\Gamma}$. Then $\overline{\Gamma}$ is analytically irreducible at O_∞.*

Proof. Suppose that $\deg \Gamma = n$. Let $(p(t), q(t)) \in K[t]^2$ be a polynomial parametrization of Γ. Using a linear automorphism we may assume that $p(t) = t^n + \cdots$, $q(t) = t^m + \cdots$, where $m < n$. Thus $O_\infty = (1 : 0 : 0)$ by Proposition 3.6. Let $f(x, y) = 0$ be the minimal equation of Γ. Multiplying $f(x, y)$ by a constant we may assume that $f(x, y) = y^n + a_1(x)y^{n-1} + \cdots + a_n(x)$, where $\deg a_k(x) < k$ for $k = 1, \ldots, n$. Let $(X : Y : Z)$ be the homogeneous coordinates in $\mathbf{P}^2(K) = K^2 \cup L_\infty$ and let $F(X, Y, Z) = Z^n f\left(\frac{X}{Z}, \frac{Y}{Z}\right)$ be the homogeneous polynomial corresponding to $f(x, y)$. Thus $F(X, Y, Z) = 0$ is the minimal homogeneous equation of $\overline{\Gamma}$. Let $u = \frac{Y}{X}, v = \frac{Z}{X}$ and let $f_\infty(u, v) = u^n + (va_1(\frac{1}{v}))^{n-1} + \cdots + v^n a_n(\frac{1}{v}) \in K[u, v]$. Then $X^{-n}F(X, Y, Z) = F\left(1, \frac{Y}{X}, \frac{Z}{X}\right) = f_\infty(u, v)$. Let $v(t) = \frac{1}{p(t^{-1})}$, $u(t) = \frac{q(t^{-1})}{p(t^{-1})}$. Then $\mathrm{ord}\, v(t) = n$, $\mathrm{ord}\, u(t) = n - m$ and $f_\infty(u(t), v(t)) = 0$. The parametrization $(v(t), u(t))$ is good since $K[p(t), q(t)] = K[t]$ implies $t^{-1} \in K[p(t^{-1}), q(t^{-1})] = K\left[\frac{1}{v(t)}, \frac{u(t)}{v(t)}\right]$. Observe that $\mathrm{ord}\, f_\infty(u, 0) = n$ and $\mathrm{ord}\, f_\infty(0, v) = \mathrm{ord}\, f_\infty = n - m$. Therefore $f_\infty(u, v)$ is irreducible in $K[[u, v]]$ by Remark 2.2. □

4. **Polynomial automorphisms of the affine plane.** In this section we prove the Jung-van der Kulk theorem on polynomial automorphisms of the plane. Following van der Kulk's ideas we study the intersection at infinity of two curves defined by the coordinates of an automorphism. There are many papers on the Jung-van der Kulk theorem (see [38, Notes on page 115]). Mostly, the authors consider only the case of zero characteristic.

A polynomial mapping $\underline{f} = (f_1, f_2) : K^2 \longrightarrow K^2$ is said to be a *polynomial automorphism* of K^2 if it is bijective and the mapping $\underline{f}^{-1} = (g_1, g_2) : K^2 \longrightarrow K^2$ is also polynomial. Clearly $\underline{f} = (f_1, f_2)$ is a polynomial automorphism if and only if $K[f_1, f_2] = K[x, y]$. All polynomial automorphisms of K^2 form a group with the composition of mappings as the group operation. This group will be denoted $GA(K^2)$.

Examples 4.1.

1. *The affine automorphisms* $(x, y) \longrightarrow (ax + by + c, a_1x + b_1y + c_1)$, *where* $ab_1 - a_1b \neq 0$.
2. *The* de Jonquières *automorphisms* $(x, y) \longrightarrow (x, y+P(x))$, *where* $P(x) \in K[x]$.

Proposition 4.2. *Let* $\underline{f} = (f_1, f_2) \in GA(K^2)$. *Then*

(1) *the polynomials* f_1, f_2 *are irreducible in* $K[x, y]$,
(2) $\operatorname{jac} \underline{f} = \frac{\partial f_1}{\partial x}\frac{\partial f_2}{\partial y} - \frac{\partial f_2}{\partial x}\frac{\partial f_1}{\partial y} = \text{constant} \neq 0$,
(3) *the affine curves* $V(f_1)$, $V(f_2)$ *are embedded lines; if* $\underline{f}^{-1} = (g_1, g_2)$ *then* $(g_1(0, v), g_2(0, v))$ *is a polynomial parametrization of* $V(f_1)$ *and* $(g_1(u, 0), g_2(u, 0))$ *is a polynomial parametrization of* $V(f_2)$.

Proof.

(1) The mapping $K[u, v] \ni g(u, v) \longrightarrow g(f_1(x, y), f_2(x, y)) \in K[x, y]$ is an isomorphism. Thus f_1 (respectively f_2) is irreducible as the image of the variable u (respectively v) by an isomorphism.
(2) Let $\underline{g} = \underline{f}^{-1}$. Then $\underline{g} \circ \underline{f} = \text{identity}$ and $((\operatorname{jac} \underline{g}) \circ \underline{f})\operatorname{jac} \underline{f} = 1$ in $K[x, y]$. Therefore $\operatorname{jac} \underline{f} = \text{constant} \neq 0$.
(3) The property follows from the identities

$$f_1(g_1(0, v), g_2(0, v)) = 0, \quad f_2(g_1(0, v), g_2(0, v)) = v,$$

$$f_1(g_1(u, 0), g_2(u, 0)) = u \text{ and } f_2(g_1(u, 0), g_2(u, 0)) = 0$$

in $K[u, v]$.

□

Jacobian Conjecture (Keller 1939) Let K be a field of characteristic zero. Suppose that $\underline{f} : K^2 \longrightarrow K^2$ is a polynomial mapping such that $\operatorname{jac} \underline{f} = \text{constant} \neq 0$. Then $\underline{f}$ is a polynomial automorphism.

This conjecture is still open, see [35]. If $\operatorname{char} K = p > 0$ then the conjecture is false: take $\underline{f}(x, y) = (x, x^p + y)$ for $(x, y) \in K^2$.

An affine algebraic curve Γ is called a *coordinate line* if there exists a polynomial automorphism of K^2 mapping it onto the axis $\{0\} \times K$. Equivalently, if there exists a polynomial automorphism $\underline{f} = (f_1, f_2)$ such that $\Gamma = V(f_1)$.

Proposition 4.3. *Any coordinate line is isomorphic to the line* K *i.e. it is an embedded line.*

Proof. Use the third part of Proposition 4.2. □

For any polynomial automorphism $\underline{f} = (f_1, f_2)$ we set $\deg \underline{f} = \max(\deg f_1, \deg f_2)$.

Proposition 4.4. *For any* $\underline{f} \in GA(K^2)$ *we get* $\deg \underline{f}^{-1} = \deg \underline{f}$.

Proof. Since f_i $(i=1,2)$ are irreducible $\deg f_i = \deg V(f_i)$ for $i=1,2$ and $\deg \underline{f} = \max(\deg V(f_1), \deg V(f_2))$. By Propositions 3.6 and 4.2 (3) we get

$$\deg V(f_1) = \max(\deg g_1(0,v), \deg g_2(0,v)) \le \max(\deg g_1, \deg g_2) = \deg \underline{f}^{-1},$$

and

$$\deg V(f_2) = \max(\deg g_1(u,0), \deg g_2(u,0)) \le \max(\deg g_1, \deg g_2) = \deg \underline{f}^{-1}.$$

Therefore $\deg \underline{f} \le \deg \underline{f}^{-1}$ for every $\underline{f} \in GA(K^2)$. Applying the inequality $\deg \underline{f} \le \deg \underline{f}^{-1}$ to $\underline{f}^{-1} \in GA(K^2)$ we get $\deg \underline{f}^{-1} \le \deg \underline{f}$. Consequently $\deg \underline{f} = \deg \underline{f}^{-1}$. □

Proposition 4.5. *(*[22]*) Let $\underline{f} \neq id$ be a polynomial automorphism. If an affine curve $\Gamma \subset K^2$ lies in the set* $\mathrm{Fix}(\underline{f}) = \{(a,b) \in K^2 \ : \ \underline{f}(a,b) = (a,b)\}$ *then $\overline{\Gamma}$ intersects L_∞ at one point.*

Proof. We may assume that $\deg \Gamma > 1$. Then $\deg f_1 > 1$ or $\deg f_2 > 1$. Suppose that $\deg f_1 > 1$. We have $\Gamma \subset V(f_1 - x)$. The curve $V(f_1 - x)$ has one point at infinity since $\deg f_1 > 1$ and $V(f_1)$ is an embedded line by Proposition 4.2 (3). Thus Γ has one point at infinity. □

Proposition 4.6. *(*[22]*) Let $\Gamma \subset K^2$ be an affine curve which intersects the line at infinity L_∞ at least in two points. Suppose that $\underline{f}, \underline{g} \in GA(K^2)$ and $\underline{f}_{|\Gamma} = \underline{g}_{|\Gamma}$. Then $\underline{f} = \underline{g}$.*

Proof. The condition $\underline{f}_{|\Gamma} = \underline{g}_{|\Gamma}$ implies that $\Gamma \subset \mathrm{Fix}(\underline{g}^{-1} \circ \underline{f})$. Since Γ has more than one point at infinity we get $\underline{g}^{-1} \circ \underline{f} = id$ and $\underline{g} = \underline{f}$ by Property 4.5. □

To prove the famous Jung-van der Kulk theorem we begin with the following basic property of polynomial automorphisms due to van der Kulk [39].

Proposition 4.7. *(*[39, Lemma on p. 36]*) Let $\underline{f} = (f_1, f_2) \in GA(K^2)$. Then of the two integers $n_1 = \deg f_1$, $n_2 = \deg f_2$ one divides the other.*

Proof. Let C_1 and C_2 be the projective curves with the affine equation $f_1(x,y) = 0$ and $f_2(x,y) = 0$. Then $\deg C_1 = n_1$, $\deg C_2 = n_2$ and each of the curves has exactly one branch at infinity. We may assume that $n_1 > 1$ and $n_2 > 1$. Then C_1 and C_2 intersect at infinity in a common point O_∞. Since C_1 and C_2 intersect in exactly one point at finite distance (with multiplicity one by Proposition 4.2 (2)), we get by Bézout's theorem $i(C_1, C_2; O_\infty) = n_1 n_2 - 1$. Obviously, we have that $i(C_i, L_\infty; O_\infty) = n_i$ for $i = 1,2$. Let $d = \gcd(n_1, n_2)$. Then by Property (vii) of the intersection multiplicity applied to the affine equations of C_1, C_2 and L_∞ in the affine system of coordinates centered at O_∞, we get $n_1 n_2 - 1 \equiv 0 \ (\mathrm{mod}\ \frac{n_1}{d} \text{ or } \frac{n_2}{d})$. This implies thad $d = n_1$ or $d = n_2$. □

Theorem 4.8. *(Jung-van der Kulk theorem) The group $GA(K^2)$ is generated by the affine and the Jonquières automorphisms.*

Proof. Let $\underline{f} = (f_1, f_2) \in GA(K^2)$. Suppose that $\underline{f}$ is not an affine automorphism. We may assume that $n_1 = \deg f_1 \leq n_2 = \deg f_2$ (if $n_1 > n_2$ then we replace $\underline{f}$ by $\underline{s} \circ \underline{f}$, where $\underline{s}(u,v) = (v,u)$). By Proposition 4.7 the rational $N = \frac{n_1}{n_2}$ is an integer. Each of the affine curves $V(f_1), V(f_2)$ has exactly one point at infinity. Since $\underline{f}$ is not an affine automorphism the points at infinity of $V(f_1)$ and $V(f_2)$ coincide. Thus we can find a constant $c \in K$ such that $\deg(f_2 - cf_1^N) < \deg f_2$. Let $\underline{t}(u,v) = (u, v - cu^N)$. Then $\underline{t}$ is a de Jonquières automorphism and $\underline{t} \circ f = (f_1, \tilde{f}_2)$, where $\deg \tilde{f}_2 < \deg f_2$. Repeating this procedure a finite number of times we get a decomposition of $\underline{f}$ into de Jonquières and affine automorphisms. □

5. **Numerical semigroups and plane branches.** The basic reference to this section is Angermüller's article [5]. The text on the planar semigroups in [33] is also very instructive. The description of the semigroups corresponding to algebroid branches was proved by Bresinsky [8] in the case of zero characteristic. The general case was proved in [5]. The irreducibility criterion due to Abhyankar and Moh (see [2], [1] and [33]) allows to test polynomials for irreducibility and works if the degree of a polynomial with respect to a distinguished variable is not divisible by the characteristic of K.

Let $\mathbf{N}$ be the set of non-negative integers. If $a_0, \ldots, a_m \in \mathbf{N}$ then $\mathbf{N}a_0 + \cdots + \mathbf{N}a_m$ stands for the set of all integers of the form $q_0a_0 + \cdots + q_ma_m$, where $q_0, \ldots, q_m \in \mathbf{N}$. A subset G of $\mathbf{N}$ closed under addition and containing 0 is called a *semigroup*. If $G = \mathbf{N}a_0 + \cdots + \mathbf{N}a_m$ then we call the sequence $(a_0, \ldots, a_m)$ system of generators of G. A semigroup G is *numerical* if $\gcd(G) = 1$.

5.1. **Nice sequences.** The following lemma is basic for further considerations.

Lemma 5.1. *Let $(v_0, \ldots, v_h)$ be a sequence of positive integers. Set $d_k = \gcd(v_0, \ldots, v_{k-1})$ for $k \in \{1, \ldots, h+1\}$ and $n_k = \frac{d_k}{d_{k+1}}$ for $k \in \{1, \ldots, h\}$. Then for every $a \in \mathbf{Z}d_{h+1}$ we have a Bézout's relation:*

$$a = a_0v_0 + a_1v_1 + \cdots + a_hv_h,$$

where $a_0, \ldots, a_h \in \mathbf{Z}$ and $0 \leq a_k < n_k$ for $k \in \{1, \ldots, h\}$. The sequence $(a_0, \ldots, a_h)$ is unique.

Proof. Existence: if $h = 0$ the lemma is obvious. Suppose that $h > 0$ and that the lemma is true for $h - 1$. Since $d_{h+1} = \gcd(d_h, v_h)$ we can write for every $a \in \mathbf{Z}d_{h+1}$: $a = a'd_h + a''v_h$ with $a', a'' \in \mathbf{Z}$. For any integer l we have $a = (a' - lv_h)d_h + (a'' + ld_h)v_h$. Thus we can take $a'' \geq 0$. Dividing a'' by $n_h = \frac{d_h}{d_{h+1}}$ we get $a'' = n_ha''' + a_h$ with $0 \leq a_h < n_h$. Therefore

$$a = a'd_h + (n_ha''' + a_h)v_h = (a'd_h + n_ha'''v_h) + a_hv_h = \left(a' + a'''\frac{v_h}{d_{h+1}}\right)d_h + a_hv_h.$$

By induction hypothesis we get $\left(a' + a''' \frac{v_h}{d_{h+1}}\right) d_h = a_0 v_0 + \cdots + a_{h-1} v_{h-1}$ with $0 \leq a_k < n_k$ for $0 \leq k \leq h-1$ and we are done.

Uniqueness: Suppose that $a_0 v_0 + \cdots + a_h v_h = a'_0 v_0 + \cdots + a'_h v_h$ with $0 \leq a_k, a'_k < n_k$ for $k > 0$. Let $a_h \leq a'_h$. Then $(a'_h - a_h) v_h \equiv 0 \mod (v_0, \ldots, v_{h-1})\mathbf{Z}$, that is $(a'_h - a_h) v_h \equiv 0 \mod (d_h)$, which implies $(a'_h - a_h) \frac{v_h}{d_{h+1}} \equiv 0 \pmod{n_h}$. Since $\frac{v_h}{d_{h+1}}$ and n_h are coprime and $0 \leq a'_h - a_h < n_h$ we get $a'_h - a_h = 0$. Uniqueness follows by induction. □

In what follows we assume that $d_{h+1} = \gcd(v_0, \ldots, v_h) = 1$. We set

$$c = \sum_{k=1}^{h} (n_k - 1) v_k - v_0 + 1$$

and we call c the *virtual conductor* of the sequence $(v_0, \ldots, v_h)$.

Proposition 5.2. *Let c be the virtual conductor of the sequence $(v_0, \ldots, v_h)$. Then $c \geq 0$ and $c = 0$ if and only if $v_k = d_{k+1}$ for all $k = 1, \ldots, h$ such that $n_k > 1$.*

Proof. Obviously $v_k \geq d_{k+1}$ for $k = 1, \ldots, h$. Therefore we get $c = \sum_{k=1}^{h} (n_k - 1) v_k - v_0 + 1 \geq \sum_{k=1}^{h} (n_k - 1) d_{k+1} - d_1 + 1 = 0$. Clearly $c = 0$ if and only if $v_k = d_{k+1}$ for all k such that $n_k > 1$. □

Proposition 5.3. *(Brauer) With the notations introduced above, if a is an integer such that $a \geq c$ then $a \in \mathbf{N} v_0 + \cdots + \mathbf{N} v_h$.*

Proof. Let's write Bézout's relation for the integer a: $a = a_0 v_0 + \cdots + a_h v_h$, where $0 \leq a_k \leq n_k - 1$ for $k = 1, \ldots, h$. Then $a_0 v_0 = a - \sum_{k=1}^{h} a_k v_k \geq c - \sum_{k=1}^{h} a_k v_k = -v_0 + 1 + \sum_{k=1}^{h} (n_k - 1 - a_k) v_k \geq -v_0 + 1$. Consequently we get $a_0 \geq \frac{-v_0+1}{v_0} = -1 + \frac{1}{v_0} > -1$, which implies $a_0 \geq 0$. □

Proposition 5.4. *Suppose that $l v_k \in \mathbf{N} v_0 + \cdots + \mathbf{N} v_{k-1}$, for an integer $l > 0$. Then $l \equiv 0 \pmod{n_k}$.*

Proof. If $l v_k \in \mathbf{N} v_0 + \cdots + \mathbf{N} v_{k-1}$ for an integer $l > 0$ then $l v_k \equiv 0 \pmod{d_k}$ and $l \frac{v_k}{d_{k+1}} \equiv 0 \pmod{n_k}$. Since $\frac{v_k}{d_{k+1}}$ and $n_k = \frac{d_k}{d_{k+1}}$ are coprime we get $l \equiv 0 \pmod{n_k}$. □

Definition 5.5. *A sequence $(v_0, \ldots, v_h)$ is* nice *if $n_k v_k \in \mathbf{N} v_0 + \cdots + \mathbf{N} v_{k-1}$ for $k = 1, \ldots, h$.*

Note that $n_1 v_1 = \left(\frac{v_1}{d_2}\right) v_0 \in \mathbf{N} v_0$. Hence the sequence (v_0, v_1) is nice. The sequence $(6, 7, 8)$ is not nice but the sequence $(6, 9, 7)$ is.

Proposition 5.6. *Let $(v_0, \ldots, v_h)$ be a nice sequence. Then for every $k \in \{1, \ldots, h\}$, $v_k \notin \mathbf{N} v_0 + \cdots + \mathbf{N} v_{k-1}$ if and only if $n_k > 1$.*

Proof. If $v_k \notin \mathbf{N}v_0 + \cdots + \mathbf{N}v_{k-1}$ then $n_k > 1$ by the definition of nice sequences. If $n_k > 1$ then $v_k \notin \mathbf{N}v_0 + \cdots + \mathbf{N}v_{k-1}$ by Property 5.4. □

Proposition 5.7. *Let $(v_0, \ldots, v_h)$ be a nice sequence and let c be the virtual conductor of $(v_0, \ldots, v_h)$. Set $G = \mathbf{N}v_0 + \cdots + \mathbf{N}v_h$. Then*

1. *if $a \in \mathbf{N}v_0 + \cdots + \mathbf{N}v_k$ then $a = a_0v_0 + \cdots + a_kv_k$ with $0 \leq a_0$ and $0 \leq a_i < n_i$ for $i = 1, \ldots, k$.*
2. *For every $a, b \in \mathbf{Z}$: if $a + b = c - 1$ then exactly one element of the pair (a, b) belongs to G.*
3. *The virtual conductor c equals the conductor of G, that is, all integers greater than or equal to c are in G and $c - 1 \notin G$.*
4. *c is an even number and $\sharp(\mathbf{N}\backslash G) = \frac{c}{2}$.*

Proof.

1. If $k = 0$ the assertion is obvious. Suppose that $k > 0$ and that the property is true for $k - 1$. By assumption we have $a = q_0v_0 + \cdots + q_kv_k$ with $q_i \geq 0$ for $i = 0, \ldots, k$. By the Euclidean division of q_k by n_k we get $q_k = q'_kn_k + a_k$ with $0 \leq a_k < n_k$. Thus $a = q_0v_0 + \cdots + q_{k-1}v_{k-1} + q'_kn_kv_k + a_kv_k = a' + a_kv_k$, where $0 \leq a_k < n_k$ and $a' \in \mathbf{N}v_0 + \cdots + \mathbf{N}v_{k-1}$ since by hypothesis the sequence $(v_0, \ldots, v_h)$ is nice so $q'_kn_kv_k \in \mathbf{N}v_0 + \cdots + \mathbf{N}v_{k-1}$. We use the induction hypothesis.
2. Take two integers $a, b \in \mathbf{Z}$ such that $a + b = c - 1$. Let us write the Bézout's relation $a = a_0v_0 + a_1v_1 + \cdots + a_hv_h$, where $a_0 \in \mathbf{Z}$ and $0 \leq a_i < n_i$ for $i \in \{1, \ldots, h\}$. Then by the definition of c we get $b = c - 1 - a = -v_0 + \sum_{k=1}^{h}(n_k - 1)v_k - a_0v_0 - \sum_{k=1}^{h} a_kv_k = -(a_0 + 1)v_0 + \sum_{k=1}^{h}(n_k - 1 - a_k)v_k$. This is a Bézout relation. To finish the proof it suffices to remark that exactly one element of the pair $(a_0, -a_0 - 1)$ is greater than or equal to zero.
3. By Property 5.3 any integer $a \geq c$ is in G. On the other hand, since $(c-1)+0 = c-1$ and $0 \in G$ we have $c-1 \notin G$, by the second statement of this proposition.
4. The mapping $[0, c-1] \cap G \ni a \to c - 1 - a \in [0, c-1] \cap (\mathbf{N}\backslash G)$ is bijective. Therefore we have $2 \cdot \sharp([0, c-1] \cap G) = c$ and the fourth claim follows.

□

Proposition 5.8. *Let $(v_0, \ldots, v_h)$ be a sequence of positive integers such that for any $a \in \mathbf{N}v_0 + \cdots + \mathbf{N}v_h$ there exist integers $a_0, a_1, \ldots, a_h$ such that $a = a_0v_0 + a_1v_1 + \cdots + a_hv_h$ and $0 \leq a_0$, $0 \leq a_k < n_k$ for $k = 1, \ldots, h$, where $n_k := \frac{d_k}{d_{k+1}}$ with $d_k = \gcd(v_0, \ldots, v_{k-1})$. Then $(v_0, \ldots, v_h)$ is a nice sequence.*

Proof. We have $n_kv_k = d_k\left(\frac{v_k}{d_{k+1}}\right) \equiv 0 \pmod{d_k}$. Therefore by Bézout's relation we get $n_kv_k = \tilde{a}_0v_0 + \tilde{a}_1v_1 + \cdots + \tilde{a}_{k-1}v_{k-1}$, where $\tilde{a}_0 \in \mathbf{Z}$, $0 \leq \tilde{a}_k < n_k$ for $k = 1, \ldots, h$. By the uniqueness of Bézout's relation $a_k = \cdots = a_h = 0$ and $n_kv_k \in \mathbf{N}v_0 + \cdots + \mathbf{N}v_{k-1}$. □

5.2. **Semigroups of plane branches.** For any irreducible power series $f \in K[[x,y]]$ we put

$$G(f) = \{i_0(f,g) \ : \ g \text{ runs over all power series such that } g \not\equiv 0 \pmod f\}.$$

Clearly $G(f)$ is a semigroup. Obviously $G(f) = G(uf)$, where u is a unit in $K[[x,y]]$. We call $G(f)$ the *semigroup associated with the branch* $\{f = 0\}$. We have $\min(G(f)\backslash\{0\}) = \operatorname{ord} f$. Hence $G(f) = \mathbf{N}$ if and only if $\operatorname{ord} f = 1$. If two branches $\{f = 0\}$ and $\{g = 0\}$ are formally equivalent then $G(f) = G(g)$. The semigroups of branches can be characterized in terms of sequences of generators. A sequence of positive integers $(r_0, \dots, r_h)$ is said to be a *characteristic sequence* if it satisfies the following two axioms

(CS1) Set $d_k = \gcd(r_0, \dots, r_{k-1})$ for $1 \le k \le h+1$. Then $d_k > d_{k+1}$ for $1 \le k \le h$ and $d_{h+1} = 1$.

(CS2) $d_k r_k < d_{k+1} r_{k+1}$ for $1 \le k < h$.

We call r_0 the *initial term* of the characteristic sequence. Remember that $n_k = \frac{d_k}{d_{k+1}}$. Hence the condition (CS2) can be rewritten in the form $n_k r_k < r_{k+1}$ for $1 \le k \le h$.

Lemma 5.9. *Any characteristic sequence is nice.*

Proof. Fix $k \in \{1, \dots, h\}$. Since $n_k r_k = d_k \left(\frac{r_k}{d_{k+1}}\right) \equiv 0 \pmod{d_k}$ we can write, by Lemma 5.1, the Bézout's relation for $a = n_k r_k$: $n_k r_k = a_0 r_0 + \cdots + a_{k-1} r_{k-1}$, where $a_0 \in \mathbf{Z}$ and $0 \le a_i < n_i$ for $i = 1, \dots, k-1$. Therefore we get $a_0 r_0 = n_k r_k - a_1 r_1 - \cdots - a_{k-1} r_{k-1} \ge n_k r_k - (n_1 - 1) r_1 - \cdots - (n_{k-1} - 1) r_{k-1} = n_k r_k - [(n_1 - 1) r_1 + \cdots + (n_{k-1} - 1) r_{k-1}] > n_k r_k - [(r_2 - r_1) + \cdots + (r_k - r_{k-1})] = n_k r_k - r_k + r_1 > 0$, which implies $a_0 > 0$. $\square$

Let $G = \mathbf{N} r_0 + \cdots + \mathbf{N} r_h$ be the semigroup generated by the characteristic sequence $(r_0, \dots, r_h)$. Then $r_k = \min(G \backslash (\mathbf{N} r_0 + \cdots \mathbf{N} r_{k-1}))$ for $1 \le k \le h$, which shows that G and r_0 determine the sequence $(r_0, \dots, r_h)$. By Lemma 5.9 and Proposition 5.7 the conductor of G is equal to $c = \sum_{k=1}^{h} (n_k - 1) r_k - r_0 + 1$.

Theorem 5.10 (Bresinsky-Angermüller Semigroup Theorem)**.**

1. *Let $f = f(x,y) \in K[[x,y]]$ be an irreducible power series. Suppose that $n = i_0(f,x) < +\infty$. Then the semigroup $G(f)$ of the branch $\{f = 0\}$ is generated by a characteristic sequence $(r_0, \dots, r_h)$, where $r_0 = n$.*
2. *Let $G \subset \mathbf{N}$ be a semigroup generated by a characteristic sequence with the initial term $n > 0$. Then there exists an irreducible power series $f = f(x,y) \in K[[x,y]]$ such that $i_0(f,x) = n$ and $G(f) = G$.*

A characteristic-blind proof of the above theorem is given in [18, Theorem 6.5].

The following result is a local version of the irreducible criterion of Abhyankar and Moh (see [1, page 99]).

Theorem 5.11 (Abhyankar-Moh irreducibility criterion)**.** *Let $f = f(x,y) \in K[[x,y]]$ be an irreducible power series such that $n = i_0(f,x) < +\infty$ and let $G(f) =$*

$\mathbf{N}r_0 + \cdots \mathbf{N}r_h$, where $r_0 = n$. If $g = g(x,y) \in K[[x,y]]$ is a power series such that $i_0(g,x) = n$ and $i_0(f,g) > d_h r_h$ then g is irreducible and $G(g) = G(f)$.

For the proof of the above theorem we refer the reader to [18, Corollary 5.8].

6. **Abhyankar-Moh theory of approximate roots and the Embedding Line Theorem.** Abhyankar in [1] gave a simplified version of [2] and [3]. Russell in [31] reproved the Abhyankar-Moh results using the Hamburger-Noether expansions with weaker assumptions on the field characteristic. A short proof of the Embedding Line Theorem due to Richman and Nowicki is included in [38] (see also [23]). Another proof was given by Ganong [16]. Suzuki [36] proved the Embedding Line Theorem independently in the case $K = \mathbf{C}$.

The Abhyankar-Moh inequality is one of the main results of [3]. Its proof given by Abhyankar and Moh relies on detailed analysis of Puiseux expansion at infinity (see also [11], Appendix A). The inequality can be also stated in terms of the semigroup associated with the branch at infinity of the given curve (see [20], [7] and [17]).

The presentation of the subject given in this section is based on [18] (see also [20] and [12]). The reader will find in [29] more references about the approximate roots.

Let R be a commutative ring with identity and let $f \in R[y]$ be a monic polynomial of degree $n > 0$. Let $d > 0$ be a divisor of n. A polynomial $g \in R[y]$ is called an *approximate dth root* of f (we will denote it $g = \sqrt[d]{f}$) if g is monic and $\deg(f - g^d) < \deg f - \deg g$. Thus we have $\deg f = \deg g^d$ and $\deg g = n/d$.

Proposition 6.1. *Let $f \in R[y]$ be a monic polynomial of degree $n > 0$. Let $d > 0$ be an integer such that d divides n. Assume that d is a unit in R. Then there exists a unique approximate dth root $\sqrt[d]{f}$ of f. If $f = y^n + a_1 y^{n-1} + \cdots + a_n$ then*

$$g = y^{n/d} + b_1 y^{(n/d)-1} + \cdots + b_{n/d},$$

where

$$b_\nu = \frac{1}{d} a_\nu + \sum_{i_1 + 2i_2 + \cdots + (\nu-1) i_{\nu-1} = \nu} \beta_{i_1,\ldots,i_{\nu-1}} a_1^{i_1} \cdots a_{\nu-1}^{i_{\nu-1}}, \tag{6.1}$$

for $\nu = 1, \ldots, \frac{n}{d}$, with $\beta_{i_1,\ldots,i_{\nu-1}} \in \mathbf{Z}\left[\frac{1}{d}\right]$ depending only on n and d.

Proof. The inequality $\deg(f - g^d) < n - \frac{n}{d}$ is equivalent to the system of equalities

$$a_\nu = d b_\nu + \sum_{i_1 + 2i_2 + \cdots + (\nu-1) i_{\nu-1} = \nu} \alpha_{i_1,\ldots,i_{\nu-1}} b_1^{i_1} \cdots b_{\nu-1}^{i_{\nu-1}}, \tag{6.2}$$

for $\nu = 1, \ldots, \frac{n}{d}$, where

$$\alpha_{i_1,\ldots,i_{\nu-1}} = \binom{d}{i_1 + \cdots + i_{\nu-1}} \frac{(i_1 + \cdots + i_{\nu-1})!}{i_1! \cdots i_{\nu-1}!}.$$

The system (6.2) of $\frac{n}{d}$ equations with the unknowns $b_1, \ldots, b_{n/d}$ has exactly one solution given by formulae (6.1). $\square$

Let $f(x,y) \in K[[x,y]]$. We say that f is a *y-distinguished polynomial* (in short: *distinguished*) if $f(x,y) = y^n + a_1(x)y^{n-1} + \cdots + a_n(x)$, where $a_i(x) \in K[[x]]$ and $a_i(0) = 0$ for $1 \le i \le n$.

Now, we can state the Abhyankar-Moh theorem on approximate roots.

Theorem 6.2 (Abhyankar-Moh Fundamental Theorem on approximate roots). *Let $f = f(x,y) \in K[[x]][y]$ be an irreducible distinguished polynomial of degree $n > 1$ with $G(f) = \mathbf{N}r_0 + \cdots + \mathbf{N}r_h$ and $r_0 = n$. Let $k \in \{1, \ldots, h\}$. Suppose that $d_k \not\equiv 0$* mod char$K$. *Then:*

1. $i_0(f, \sqrt[d_k]{f}) = r_k$,
2. $\sqrt[d_k]{f}$ *is an irreducible distinguished polynomial of degree n/d_k and $G(\sqrt[d_k]{f}) = \mathbf{N}\frac{r_0}{d_k} + \mathbf{N}\frac{r_1}{d_k} + \cdots + \mathbf{N}\frac{r_{k-1}}{d_k}$.*

For the proof of the above theorem we refer the reader to [18].

Let Γ be an affine irreducible curve in K^2. We say that Γ *has one branch at infinity* if the projective closure $\overline{\Gamma}$ of Γ intersects the line at infinity L_∞ in only one point O_∞, and $\overline{\Gamma}$ is analytically irreducible at O_∞.

Let (u,v) be a system of affine coordinates centered at O_∞ such that L_∞ has the equation $v = 0$. Let $f_\infty(u,v) = 0$ be a polynomial equation of $\overline{\Gamma}$ of total degree $n = \deg\Gamma$. Multiplying f_∞ by a constant we may assume that f_∞ is a u-distinguished polynomial (of degree n since $\overline{\Gamma}$ and L_∞ intersect only at O_∞) irreducible in $K[[u,v]]$.

Let $G(f_\infty) = \mathbf{N}r_0 + \cdots + \mathbf{N}r_h$, with $r_0 = n = \deg\Gamma$. We call $(r_0, r_1, \ldots, r_h)$ the *characteristic of* Γ *at infinity.* Clearly it is independent on the system of coordinates. Observe that $r_1 = \operatorname{ord} f_\infty = \operatorname{mult}_{O_\infty}\overline{\Gamma}$. Let $n' = \operatorname{mult}_{O_\infty}\overline{\Gamma}$. We call Γ *permissible* if $d := \gcd(n, n') \not\equiv 0$ (mod char K).

Theorem 6.3 (Abhyankar-Moh inequality). *Assume that Γ is an affine curve of degree $n > 1$ with one branch at infinity and let $(r_0, \ldots, r_h)$ be the characteristic of Γ at infinity. If Γ is permissible, then $d_h r_h < n^2$.*

Proof. Let (u,v) be the affine coordinate system introduced above and let $f_\infty(u,v) = 0$ be the affine equation of $\overline{\Gamma}$. Then $G(f_\infty) = \mathbf{N}r_0 + \cdots + \mathbf{N}r_h$ with $r_0 = n$ and $r_1 = n'$. Therefore $d_2 = \gcd(r_0, r_1) \not\equiv 0$ (mod char K) and consequently $d_h \not\equiv 0$ (mod char K).

By Theorem 6.2 applied in the case $k = h$ the approximate root (with respect to u) $\sqrt[d_h]{f_\infty}$ exists and $i_0(f_\infty, \sqrt[d_h]{f_\infty}) = r_h$. The total degree of $\sqrt[d_h]{f_\infty}$ is $\frac{n}{d_h}$ by formulae (6.1) of Proposition 6.1. Thus by Bézout's theorem applied to f_∞ and $\sqrt[d_h]{f_\infty}$ we get $r_h = i_0(f_\infty, \sqrt[d_h]{f_\infty}) \le n\frac{n}{d_h}$. In fact, we have $r_h < n\frac{n}{d_h}$ for $r_h = n\frac{n}{d_h}$ would imply $r_h \equiv 0$ (mod d_h) which is impossible. $\square$

Now we can state

Theorem 6.4 (Abhyankar-Moh Embedding Line Theorem, first formulation)**.** *Assume that* Γ *is an embedded line of degree* $n > 1$ *and let* $n' = \mathrm{mult}_{O_\infty}\overline{\Gamma}$*. Suppose that* Γ *is permissible. Then* $n - n'$ *divides* n*.*

To prove Theorem 6.4 we need the Abhyankar-Moh inequality and two lemmas.

Lemma 6.5. *Let* Γ *be an embedded line of degree* $n > 1$ *and let* $(r_0, \dots, r_h)$ *with* $r_0 = n$ *be the characteristic of* Γ *at infinity. Then* $\sum_{k=1}^{h}(n_k - 1)r_k = (r_0 - 1)^2$.

Proof. Let c be the conductor of the semigroup $\mathbf{N}r_0 + \cdots + \mathbf{N}r_h$. By the genus formula applied to $\overline{\Gamma}$ we get $(n-1)(n-2) = 2\delta$, where δ is the double point number at O_∞ (for the definition of δ and the genus formula we refer the reader to [24, Definition 14.8, Theorem 14.7]). Since $c = 2\delta$ we get the lemma from the conductor formula $c = \sum_{k=1}^{h}(n_k - 1)r_k - r_0 + 1$. □

Lemma 6.6. *Let* $(r_0, \dots, r_h)$ *be a characteristic sequence such that*

(a) $\sum_{k=1}^{h}(n_k - 1)r_k = (r_0 - 1)^2$,

(b) $d_h r_h < r_0^2$.

Then $r_k = \frac{d_1^2}{d_k} - d_{k+1}$*, for* $k \in \{1, \dots, h\}$*.*

Proof. Let $q_k := \frac{n^2}{d_k d_{k+1}} - \frac{r_k}{d_{k+1}}$ for $k \in \{1, \dots, h\}$. Then q_k is an integer and $q_k := \frac{n^2 - d_k r_k}{d_k d_{k+1}} = \frac{r_0^2 - d_k r_k}{d_k d_{k+1}} > 0$ by condition (b). Hence $q_k \geq 1$ and $\frac{n^2}{d_k} - r_k = d_{k+1} q_k \geq d_{k+1}$, which implies

$$\frac{n^2}{d_k} - d_{k+1} - r_k \geq 0 \quad \text{for } k = 1, \dots, h. \tag{6.3}$$

On the other hand

$$\begin{aligned} \sum_{k=1}^{h}(n_k - 1)\left(\frac{n^2}{d_k} - d_{k+1} - r_k\right) &= \sum_{k=1}^{h}(n_k - 1)\left(\frac{n^2}{d_k} - d_{k+1}\right) - \sum_{k=1}^{h}(n_k - 1)r_k \\ &= (n-1)^2 - (n-1)^2 = 0. \end{aligned} \tag{6.4}$$

Combining (6.3) and (6.4) we get $r_k = \frac{n^2}{d_k} - d_{k+1}$, for $k \in \{1, \dots, h\}$. □

Proof. **(of Theorem 6.4)** Let $(r_0, \dots, r_h)$, $r_0 = n$ be the characteristic of Γ at infinity. By Theorem 6.3 and Lemma 6.5 the characteristic sequence $(r_0, \dots, r_h)$ verifies conditions (a) and (b) of Lemma 6.6 and consequently $r_k = \frac{d_1^2}{d_k} - d_{k+1}$, for $k = 1, \dots, h$. In particular, $r_1 = d_1 - d_2 = r_0 - d_2$. Hence $n - n' = r_0 - r_1 = d_2 = \gcd(r_0, r_1) = \gcd(n, n')$ and we are done. □

Theorem 6.7 (Abhyankar-Moh Embedding Line Theorem, second formulation)**.** *If* $(p,q) : K \longrightarrow K^2$ *is a polynomial embedding such that* $n = \deg p$*,* $m = \deg q > 0$ *and* $\gcd(m,n) \not\equiv 0 \pmod{\operatorname{char} K}$ *then* m *divides* n *or* n *divides* m*.*

Proof. We may assume that $1 < m < n$. Let Γ be an embedded line with polynomial parametrization (p,q), that is $\Gamma = (p,q)(K)$. Then $O_\infty = (1:0:0)$, Γ is of degree n and its multiplicity at O_∞ is $n' = n - m$ by Proposition 3.6. Therefore Γ is permissible. Apply Theorem 6.4 to the curve Γ. □

Corollary 6.8. *Let* K *be of characteristic zero. Then any embedded line is a coordinate line.*

Proof. Let $\Gamma \subset K^2$ be an embedded line. Proceeding by induction on $\deg \Gamma$ if suffices to check that if $\deg \Gamma > 1$ then there exists an automorphism $\underline{f} \in GA(K^2)$ such that $\deg \underline{f}(\Gamma) < \deg \Gamma$ (the image of an embedded line by a polynomial automorphism is again an embedded line). Let $n = \deg \Gamma > 1$. Using a linear automorphism we may assume that $\Gamma = (p,q)(K^2)$, where $p(t) = t^n + \cdots$ and $q(t) = t^m + \cdots$, with $n > m$ and the dots mean terms in t of degree less than n (respectively m). By Theorem 6.7 the number $N = \frac{n}{m}$ is an integer. Let $\underline{f}(u,v) = (u - v^N, v)$. Then $\underline{f} \in GA(K^2)$ and $\underline{f}(\Gamma)$ is an embedded line with parametrization $(p(t) - q(t)^N, q(t))$. Therefore we have that $\deg \underline{f}(\Gamma) = \max\{\deg(p(t) - q(t)^N), \deg q(t)\} < n = \deg \Gamma$. □

Remark 6.9. The condition $\gcd(m,n) \not\equiv 0 \pmod{\operatorname{char} K}$ is automatically satisfied for $\operatorname{char} K = 0$, and is essential for $l = \operatorname{char} K \neq 0$. The following example is due to Nagata (see [26] and [17]). Let $\operatorname{char} K = l > 0$ and let $a > 1$ be an integer coprime with l. Consider the polynomials $p(t) = t^{l^2}$, $q(t) = t + t^{al}$. Then for $f(x,y) = (y^l - x^a)^l - x$ and $g(x,y) = y - (y^l - x^a)^a$ we have $f(p(t), q(t)) = 0$ and $g(p(t), q(t)) = t$, which shows that the affine curve $f(x,y) = 0$ is an embedded line. Moreover neither $\deg p$ divides $\deg q$ nor $\deg q$ divides $\deg p$. ◇

7. **Curves with one branch at infinity and the Abhyankar-Moh Semigroup Theorem.** Our presentation of the Abhyankar-Moh Semigroup Theorem follows the treatment of Pinkham [27]. In [32] the notion of *planar semigroup* is introduced and studied. In [33] and [30] the construction of a curve with given planar semigroup is given. The reader will find in [40] an interesting application of the Abhyankar-Moh Semigroup Theorem. For further developments of the theory of the curves with one branch at infinity we refer the reader to [4], [6], [10], [13], [14], [15]. The pencil theorem is due to [25] and [13].

An affine irreducible curve $\Gamma \subset K^2$ has exactly one point at infinity if its projective closure $\overline{\Gamma}$ intersects the line at infinity L_∞ exactly in one point. Suppose that $\overline{\Gamma}$ intersects L_∞ in $O_\infty = (1:0:0)$. Then Γ has a minimal affine equation of the form $f(x,y) = y^n + a_1(x)y^{n-1} + \cdots + a_n(x)$, where $\deg a_k(x) < k$ for $k = 1, \ldots, n = \deg \Gamma$. Let $(X : Y : Z)$ be the homogeneous coordinates in $\mathbf{P}^2(K) = K^2 \cup L_\infty$ and let $F(X,Y,Z) = Z^n f\left(\frac{X}{Z}, \frac{Y}{Z}\right)$ be the homogeneous polynomial corresponding to $f(x,y)$. Thus $F(X,Y,Z) = 0$ is the minimal homogeneous equation of $C := \overline{\Gamma}$. Let

$u=\frac{Y}{X}$, $v=\frac{Z}{X}$ and let $f_\infty(u,v)=u^n+va_1\left(\frac{1}{v}\right)u^{n-1}+\cdots+v^na_n\left(\frac{1}{v}\right)\in K[u,v]$. Then $f_\infty(u,v)$ is a distinguished polynomial; the equation describes the portion of C lying in $\mathbf{P}^2(K)\backslash V(X)$. The curve Γ has one branch at infinity if and only if $f_\infty(u,v)$ is irreducible in $K[[u,v]]$. We have $i_0(f_\infty,v)=n=\deg C$ and $i_0(f_\infty,u)=n'=\mathrm{mult}_{O_\infty}C$ if C is unitangent at O_∞. Clearly

$$X^{-n}F(X,Y,Z)=F\left(1,\frac{Y}{X},\frac{Z}{X}\right)=f_\infty\left(\frac{Y}{X},\frac{Z}{X}\right).$$

Lemma 7.1. *Let $g=g(x,y)=y^n+b_1(x)y^{n-1}+\cdots+b_n(x)\in K[x,y]$, $\deg b_k(x)<k$ for $k=1,\dots,n$. Suppose that the polynomial $g_\infty(u,v)=u^n+vb_1\left(\frac{1}{v}\right)u^{n-1}+\cdots+v^nb_n\left(\frac{1}{v}\right)\in K[u,v]$ is irreducible in $K[[u,v]]$. Then $g=g(x,y)$ is irreducible in $K[x,y]$.*

Proof. Suppose that $g=g(x,y)$ is not irreducible in $K[x,y]$. Then $g=g_1g_2$, where $g_i=y^{n_i}+$ terms of degree $<n_i$ for $i=1,2$ and $g_\infty=(g_1)_\infty(g_2)_\infty$, which is a contradiction since the factors $(g_i)_\infty$, $i=1,2$, are u-distinguished polynomials in $K[[u,v]]$. □

Theorem 7.2 (Moh-Ephraim Pencil Theorem). *If $f(x,y)\in K[x,y]$ is an irreducible polynomial such that the affine curve $\Gamma: f(x,y)=0$ has one branch at infinity of characteristic $(r_0,r_1,\dots,r_h)$, $r_0=n=\deg\Gamma$ and is permissible then the polynomials $f_\lambda(x,y)=f(x,y)-\lambda$, $\lambda\in K$ are irreducible in $K[x,y]$ and the curves $\Gamma_\lambda: f_\lambda(x,y)=0$ have one branch at infinity of characteristic $(r_0,r_1,\dots,r_h)$.*

Proof. Let $\lambda\neq 0$. The curves $f_\lambda=0$ and $f=0$ do not intersect in K^2 and we get $i_0((f_\lambda)_\infty,f_\infty)=\deg f_\lambda\cdot\deg f=n^2>d_hr_h$ by Bézout's theorem and the Abhyankar-Moh inequality. Now, the Abhyankar-Moh irreducible criterion (Theorem 5.11) implies that $(f_\lambda)_\infty$ is irreducible in $K[[u,v]]$ of characteristic $(r_0,r_1,\dots,r_h)$. To finish the proof use Lemma 7.1. □

Let us keep the notations and assumptions introduced at the beginning of this section. Let $f(x,y)=0$ be the minimal equation of an affine irreducible curve Γ with one branch at infinity. Define

$$\deg_f(g)=\sum_{P\in K^2}i(f,g;P)$$

for any polynomial g such that f does not divide g in $K[x,y]$. Clearly $\deg_f(gg')=\deg_f(g)+\deg_f(g')$ and the subset of $\mathbf{N}$ defined by

$$W(f):=\{\deg_f(g)\in\mathbf{N}\ :\ g\not\equiv 0 \bmod (f)\}$$

is a semigroup. Observe that $\deg_f(x)=n$ and $\deg_f(y)=n-n'$.

Lemma 7.3. $\deg_f(g+g')\le\max\{\deg_f(g),\deg_f(g')\}$ *provided that f does not divide gg'.*

Proof. Let $O_\infty = (1:0:0)$ be the unique point at infinity of Γ and let $u = \frac{Y}{X}$, $v = \frac{Z}{X}$ be the affine coordinates centered at O_∞. Let $f_\infty(u,v) = 0$ and $g_\infty(u,v) = 0$ be the affine equations of the projective curves $F = 0$ and $G = 0$. Since $G(X,Y,Z)$ is the homogeneous form corresponding to $g(x,y)$ then $g(x,y) = G(x,y,1)$ and $\frac{G(X,Y,Z)}{X^m} = g_\infty\left(\frac{Y}{X}, \frac{Z}{X}\right)$, where $m = \deg g = \deg G$. Let $(u(t), v(t))$ be a good parametrization of the algebroid curve $f_\infty(u,v) = 0$. Then, by Bézout's theorem

$$\begin{aligned}
\deg_f(g) &= \deg f \deg g - i_0(f_\infty, g_\infty) = \text{ord } (v(t))^m - \text{ord } g_\infty(u(t), v(t)) \\
&= -\text{ord } \frac{g_\infty(u(t), v(t))}{(v(t))^m} = -\text{ord } \frac{G(1, u(t), v(t))}{(v(t))^m} = -\text{ord } G\left(\frac{1}{v(t)}, \frac{u(t)}{v(t)}, 1\right) \\
&= -\text{ord } g\left(\frac{1}{v(t)}, \frac{u(t)}{v(t)}\right)
\end{aligned}$$

and the lemma follows from the last equality. □

Now, let us state

Theorem 7.4 (Abhyankar-Moh Semigroup Theorem). *Let Γ be an irreducible affine curve with one branch at infinity of degree $n > 1$. Assume that Γ is permissible and let $(r_0, \dots, r_h)$ be the characteristic of Γ at infinity with $r_0 = n = \deg \Gamma$. Let $\delta_0 = r_0$, $\delta_k = \frac{n^2}{d_k} - r_k$, for $k = 1, \dots, h$. Then*

1. *The integers $\delta_0, \dots, \delta_h$ are positive and* $\gcd(\delta_0, \dots, \delta_{k-1}) = d_k$ *for $k = 1, \dots, h+1$.*
2. *$\delta_1 < \delta_0$ and $\delta_k < n_{k-1}\delta_{k-1}$ for $k = 2, \dots, h$.*
3. *$n_k\delta_k \in \mathbf{N}\delta_0 + \cdots + \mathbf{N}\delta_{k-1}$ for $k = 1, \dots, h$.*
4. *$W(f) = \mathbf{N}\delta_0 + \cdots + \mathbf{N}\delta_h$.*

Proof. The first and the second statement follow from the Abhyankar-Moh inequality which implies $\delta_k > 0$ for $k = 1, \dots, h$ and from the corresponding properties of the sequence $r_0, \dots, r_h$.

In order to prove the third and fourth statements, let us return to the notations and assumptions from the beginning of this section. In particular, the minimal polynomial of Γ is of the form $f(x,y) = y^n + a_1(x)y^{n-1} + \cdots + a_n(x)$, where $\deg a_k(x) < k$ for $k = 1, \dots, n$ and $f_\infty(u,v) = u^n + va_1\left(\frac{1}{v}\right)u^{n-1} + \cdots + v^n a_n\left(\frac{1}{v}\right)$ in the coordinates $u = \frac{y}{x}$, $v = \frac{1}{x}$. If d is invertible in K then the approximate roots $\sqrt[d]{f}$ (with respect to y) and $\sqrt[d]{f_\infty}$ (with respect to u) exist and $\sqrt[d]{f_\infty} = \left(\sqrt[d]{f}\right)_\infty$ by equality (6.1) in Proposition 6.1. Assume that Γ is permissible, that is, $d_2 \equiv \gcd(r_0, r_1) = \gcd(n, n') \not\equiv 0 \pmod{\text{char } K}$. Then the approximate roots $\sqrt[d_k]{f}$, $k = 2, \dots, h$ exist and again by equality (6.1) in Proposition 6.1, the total degree of $\sqrt[d_k]{f}$ is $\frac{n}{d_k}$. □

Lemma 7.5. *Suppose that $d_k \not\equiv 0$ mod char K for some $k \in \{1, \dots, h\}$. Then* $\deg_f\left(\sqrt[d_k]{f}\right) = \delta_k$ *for $k = 1, \dots, h$.*

Proof. By Bézout's theorem and the Abhyankar-Moh Theorem on approximate roots we get

$$\begin{aligned}\deg_f\left(\sqrt[d_k]{f}\right) &= \deg f\cdot\deg\sqrt[d_k]{f}-i_0\left(f_\infty,\left(\sqrt[d_k]{f}\right)_\infty\right)=\frac{n^2}{d_k}-i_0\left(f_\infty,\sqrt[d_k]{f_\infty}\right)\\ &= \frac{n^2}{d_k}-r_k=\delta_k.\end{aligned}$$

□

Since $d_2 \not\equiv 0 \pmod{\operatorname{char} K}$ we have $d_k \not\equiv 0 \pmod{\operatorname{char} K}$ for $k \geq 2$ and the approximate roots $\sqrt[d_k]{f}$ exist for $k \geq 2$.

Lemma 7.6. *Let $f_1 = y$ and $f_k = \sqrt[d_k]{f}$ for $k = 2, \dots, h$. Any polynomial $g \in K[x, y]$ of y-degree strictly less than n has a unique expansion of the form*

$$g = \sum g_{\alpha_1,\dots,\alpha_h} (f_1)^{\alpha_1} \cdots (f_h)^{\alpha_h},$$

where $g_{\alpha_1,\dots,\alpha_h} \in K[x]$ and $0 \leq \alpha_1 < n_1, \dots, 0 \leq \alpha_h < n_h$. Moreover,

1. *The y-degrees of the terms appearing in the right-hand side of the preceding equality are all distinct.*
2. *The degrees with respect to f*

$$\deg_f\left(g_{\alpha_1,\dots,\alpha_h}(f_1)^{\alpha_1}\cdots(f_h)^{\alpha_h}\right) = \alpha_0\delta_0 + \cdots + \alpha_h\delta_h, \quad \text{where } \alpha_0 = \deg g_{\alpha_1,\dots,\alpha_h},$$

are pairwise distinct.

Proof. The existence and uniqueness of the expansion and the inequality of the y-degrees holds for polynomials with coefficients in arbitrary integral domain (see [1, Section 2]). The degrees with respect to f are pairwise distinct by the uniqueness of Bézout's relation. □

Let $g \in K[x, y]$ and let $\overline{g} \in K[x, y]$, $\deg_y \overline{g} < n$ be the remainder of the Euclidean division of g by f. Then $\deg_f g = \deg_f \overline{g}$. Therefore, by Lemma 7.6, we get

$$\begin{aligned}W(f) &= \{\deg_f g \ :\ \deg_y g < n \text{ and } g \neq 0\}\\ &= \{\alpha_0\delta_0 + \cdots + \alpha_h\delta_h \ :\ 0 \leq \alpha_0 \ \text{ and } 0 \leq \alpha_k < n_k \text{ for } k = 1, \dots, h\}.\end{aligned}$$

Therefore $W(f) = \mathbf{N}\delta_0 + \cdots + \mathbf{N}\delta_h$ and the sequence $\delta_0, \dots, \delta_h$ is nice by Proposition 5.8.

Using the Abhyankar-Moh Semigroup Theorem we can prove the Embedding Line without resorting to the genus formula. Let $\Gamma \subseteq K^2$ be an embedded line. Set $n = \deg\Gamma$, $n' = \operatorname{mult}_{O_\infty}\overline{\Gamma}$ and assume that Γ is permissible. Let $f(x, y) = 0$ be the minimal equation of Γ and let $(p(t), q(t)) \in K[t]^2$ be the polynomial parametrization of Γ. One checks that $\deg_f(g) = \deg g(p(t), q(t))$ (see the proof of Proposition 3.5) and consequently

$$W(f) = \{\deg g(p(t), q(t)) \ :\ g(x, y) \in K[x, y]\ ,\ g(p(t), q(t)) \neq 0\}.$$

Since Γ is an embedded line there exists a polynomial $g(x, y)$ such that $g(p(t), q(t)) = t$. Therefore $1 \in W(f)$, $W(f) = \mathbf{N}$ and the conductor γ of $W(f)$ is equal to 0. Since the sequence $\delta_0, \ldots, \delta_h$ is nice, we have by the third part of Proposition 5.7 that $\gamma = \sum_{k=1}^{h}(n_k - 1)\delta_k - \delta_0 + 1$. By Property 5.2 $\gamma = 0$ implies $\delta_k = d_{k+1}$. In particular, $\delta_1 = d_2 = \gcd(\delta_0, \delta_1)$ and δ_1 divides δ_0, which proves Theorem 6.4 since $\delta_0 = n$ and $\delta_1 = n - n'$.

A numerical semigroup G generated by a sequence of positive integers $\delta_0, \ldots, \delta_h$ is called *planar* if it verifies

1. If $d_k = \gcd(\delta_0, \ldots, \delta_{k-1})$ for $1 \leq k \leq h+1$ and $n_k = \frac{d_k}{d_{k+1}}$, $1 \leq k \leq h$, then $d_{h+1} = 1$ and $n_k > 1$ for $1 \leq k \leq h$.
2. For $1 \leq k \leq h$, $n_k\delta_k$ belongs to $\mathbf{N}\delta_0 + \cdots + \mathbf{N}\delta_{k-1}$.
3. $\delta_k < n_{k-1}\delta_{k-1}$ for $k = 2, \ldots, h$.

The semigroup $W(f)$ termed also *Weierstrass semigroup* is planar.

REFERENCES

[1] Abhyankar, S.S., *Expansion techniques in Algebraic Geometry,* Tata Institute of Fundamental Research Lectures on Mathematics and Physics, 57, Tata Institute of Fundamental Research, Bombay, iv+168 pp., 1977.

[2] Abhyankar, S.S. and Moh, T.T., Newton-Puiseux expansion and generalized Tschirnhausen transformation I, II, *J. reine angew. Math.*, 260, 47–83; ibid. 261, 29–54, 1973.

[3] Abhyankar, S.S. and Moh, T.T., Embeddings of the line in the plane, *J. reine angew. Math.*, 276, 148–166, 1975.

[4] Abhyankar, S.S. and Sathaye, A., Uniqueness of plane embeddings of special curves, *Proc. Amer. Math. Soc.*, 124, no. 4, 1061–1069, 1996.

[5] Angermüller, G., Die Wertehalbgruppe einer ebenen irreduziblen algebroiden Kurve, *Math. Z.*, 153, no. 3, 267–282, 1977.

[6] Assi, A. and García-Sánchez, P.A., Algorithms for curves with one place at infinity, *Journal of Symbolic Computation*, 74, 475–492, 2016.

[7] Barrolleta, R.D., García Barroso, E.R. and Płoski, A., On the Abhyankar-Moh inequality, *Universitatis Iagellonicae Acta Mathematica,* LII, 7–14, 2015.

[8] Bresinsky, H., Semigroups corresponding to algebroid branches in the plane, *Proc. Amer. Math. Soc.*, 32, no. 2, 381–384, 1972.

[9] Campillo, A., *Algebroid curves in positive characteristic,* Lecture Notes in Mathematics, 813. Springer Verlag, Berlin, v+168 pp., 1980.

[10] Campillo, A. and Farrán, J.I., Computing Weierstrass Semigroups and the Feng-Rao Distance from Singular Plane Models, *Finite Fields and Their Applications,* 6, 71–92, 2000.

[11] Casas-Alvero, E., *Singularities of plane curves,* London Mathematical Society Lecture Note Series, 276, Cambridge University Press, Cambridge, xvi+345 pp., 2000.

[12] Chang, H. C. and Wang, L. C., An intersection-theoretical proof of the embedding line theorem, *J. Algebra,* 161, no. 2, 467–479, 1993.

[13] Ephraim, R., Special polars and curves with one place at infinity, *Singularities, Part 1 (Arcata, Calif., 1981),* Proc. Sympos. Pure Math., 40, Amer. Math. Soc., Providence, RI, 353–359, 1983.

[14] Fujimoto, M. and Suzuki, M., Construction of affine plane curves with one place at infinity, *Osaka J. Math.,* 39 no. 4, 1005–1027, 2002.

[15] Fujimoto, M., Suzuki, M. and Yokoyama, K., On polynomial curves in the affine plane, *Osaka J. Math.,* 43, no. 3, 597–608, 2006.

[16] Ganong, R., On plane curves with one place at infinity, *J. reine angew. Math.,* 307/308, 173–193, 1979.

[17] García Barroso, E.R., Gwoździewicz, J. and Płoski, A., Semigroups corresponding to branches at infinity of coordinate lines in the affine plane, *Semigroup Forum,* 92, 3, 534–540, 2016.

[18] García Barroso, E.R. and Płoski, A., An approach to plane algebroid branches, *Revista Matemática Complutense,* 28, 1, 227–252, 2015.

[19] García Barroso, E.R. and Płoski, A., On the intersection multiplicity of plane branches, *arXiv: 1710.05346, to appear in Colloq. Math.,* 2017.

[20] Gwoździewicz, J. and Płoski,A., On the approximate roots of polynomials, *Ann. Polon. Math.,* 60, no. 3, 199–210, 1995.

[21] Hirschfeld, J.W.P., Korchmáros, G. and Torres, F., *Algebraic curves over a finite field.* Princeton Series in Applied Mathematics, Princeton University Press, Princeton, NJ, xx+696 pp., 2008.

[22] Jelonek, Z., Sets determining polynomial automorphism of $\mathbf{C}^2$, *Bull. Polish Acad. Sci. Math.,* 37, no. 1–6, 247–250, 1989.

[23] Kang, M-Ch., On Abhyankar Moh's epimorphism theorem, *Amer. J. of Math.,* 113, 399–421, 1991.

[24] Kunz, E., *Introduction to Plane Algebraic Curves,* Translated from the 1991 German edition by Richard G. Belshoff. Birkhäuser Boston, Inc., Boston, MA, xiv+293 pp., 2005.

[25] Moh, T.T., On analytic irreducibility at ∞ of a pencil of curves, *Proc. Amer. Math. Soc.,* 44, 22–24, 1974.

[26] Nagata, M., A theorem of Gutwirth, *J. Math. Kyoto Univ.,* 11, 149–154, 1971.

[27] Pinkham, H., *Courbes planes ayant une seule place a l'infini,* Séminaire sur les Singularités des surfaces, Centre de Mathématiques de l'École Polytechnique, Année 1977–1978.

[28] Płoski, A., Introduction to the local theory of plane algebraic curves, *Analytic and Algebraic Geometry,* Eds. T. Krasiński and S. Spodzieja. Łódź University Press, 115–134, 2013.

[29] Popescu-Pampu, P., Approximate roots, *Valuation theory and its applications,* Vol. II (Saskatoon, SK, 1999), Fields Inst. Commun., 33, Amer. Math. Soc., Providence, RI, 285–321, 2003.

[30] Reguera López, A., Semigroups and clusters at infinitiy, Algebraic geometry and singularities (La Rábida, 1991), *Progr. Math.,* 134, Birkhäuser, Basel, 339–374, 1996.

[31] Russell, P., Hamburger-Noether expansions and approximate roots of polynomials, *Manuscripta Math.,* 31, no. 1–3, 25–95, 1980.

[32] Sathaye, A., On planar curves, *Amer. J. Math.,* 99, no. 5, 1105–1135, 1977.

[33] Sathaye, A. and Stenerson, J., Plane polynomial curves, *Algebraic geometry and its applications,* West Lafayette, IN, 1990, Springer, New York, 121–142, 1994.

[34] Seidenberg, A., *Elements of the theory of algebraic curves,* Addison-Wesley Publishing Co., Reading, Mass.- London-Don Mills, Ont. viii+216 pp., 1968.

[35] Smale, S., Mathematical Problems for the Next Century, *Mathematical Intelligencer*, vol. 20, Springer, 7–15, 1998.

[36] Suzuki, M., Propriétés topologiques des polynômes de deux variables complexes et automorphisms algébriques de l'espace $\mathbf{C}^2$, J. Math. Soc. Japan, 26, 3, 241–257, 1974.

[37] Suzuki, M., Affine plane curves with one place at infinity, *Ann. Inst. Fourier* (Grenoble), 49, no. 2, 375–404, 1999.

[38] van den Essen, A., *Polynomial automorphisms and the jacobian conjecture,* Progress in Mathematics, vol. 190. Birkhäuser, 2000.

[39] van der Kulk, W., On polynomial rings in two variables, *Nieuw Arch. Wiskunde,* 1, 3, 33–41, 1953.

[40] Xu, Y., A strong Abhyankar-Moh theorem and criterion of embedded line, *J. Algebra,* 409, 382–386, 2014.

E-mail address: ergarcia@ull.es; matap@tu.kielce.pl