On the Milnor Formula in Arbitrary Characteristic

Evelia R. García Barroso and Arkadiusz Płoski

Dedicated to Antonio Campillo on the occasion of his 65th birthday

Abstract The Milnor formula $\mu = 2\delta - r + 1$ relates the Milnor number μ, the double point number δ and the number r of branches of a plane curve singularity. It holds over the fields of characteristic zero. Melle and Wall based on a result by Deligne proved the inequality $\mu \geq 2\delta - r + 1$ in arbitrary characteristic and showed that the equality $\mu = 2\delta - r + 1$ characterizes the singularities with no wild vanishing cycles. In this note we give an account of results on the Milnor formula in characteristic p. It holds if the plane singularity is Newton non-degenerate (Boubakri et al. Rev. Mat. Complut. 25:61–85, 2010) or if p is greater than the intersection number of the singularity with its generic polar (Nguyen Annales de l'Institut Fourier, Tome 66(5):2047–2066, 2016). Then we improve our result on the Milnor number of irreducible singularities (Bull. Lond. Math. Soc. 48:94–98, 2016). Our considerations are based on the properties of polars of plane singularities in characteristic p.

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1 Introduction

John Milnor proved in his celebrated book [17] the formula

$$\mu = 2\delta - r + 1, \tag{M}$$

where μ is the Milnor number, δ the double point number and r the number of branches of a plane curve singularity. The Milnor's proof of (M) is based on topological considerations. A proof given by Risler [21] is algebraic and shows that (M) holds in characteristic zero.

On the other hand Melle and Wall based on a result by Deligne [5] proved the inequality $\mu \geq 2\delta - r + 1$ in arbitrary characteristic and showed that the Milnor formula holds if and only if the singularity has not wild *vanishing cycles* [16]. In the sequel we will call a *tame singularity* any plane curve singularity verifying (M).

Recently some papers on the singularities satisfying (M) in characteristic p appeared. In [1] the authors showed that planar Newton non-degenerate singularities are tame. Different notions of non-degeneracy for plane curve singularities are discussed in [10]. In [18] the author proved that if the characteristic p is greater than the kappa invariant then the singularity is tame. In [7] and [11] the case of irreducible singularities is investigated. Our aim is to give an account of the above-mentioned results.

In Sect. 2 we prove that any semi-quasihomogeneous singularity is tame. Our proof is different from that given in [1] and can be extended to the case of Kouchnirenko nondegenerate singularities ([1, Theorem 9]). In Sects. 3 and 4 we generalize Teissier's lemma ([22, Chap. II, Proposition 1.2]) relating the intersection number of the singularity with its polar and the Minor number to the case of arbitrary characteristic and reprove the result due to H.D. Nguyen [18, Corollary 3.2] in the following form: if $p > \mu(f) + \mathrm{ord}(f) - 1$ then the singularity is tame.

Section 5 is devoted to the strengthened version of our result on the Milnor number of irreducible singularities.

2 Semi-quasihomogeneous Singularities

Let $\mathbf{K}$ be an algebraically closed field of characteristic $p \geq 0$. For any formal power series $f \in \mathbf{K}[[x, y]]$ we denote by $\mathrm{ord}(f)$ (resp. $\mathrm{in}(f)$) the *order* (resp. the *initial form* of f). A power series $l \in \mathbf{K}[[x, y]]$ is called a *regular parameter* if $\mathrm{ord}(l) = 1$. A *plane curve singularity* (in short: *a singularity*) is a nonzero power series f of order greater than one. For any power series $f, g \in \mathbf{K}[[x, y]]$ we put $i_0(f, g) := \dim_{\mathbf{K}} \mathbf{K}[[x, y]]/(f, g)$ and called it the *intersection number* of f and g. The *Milnor number* of f is

$$\mu(f) := \dim_{\mathbf{K}} \mathbf{K}[[x, y]]/\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right).$$

If Φ is an automorphism of $\mathbf{K}[[x, y]]$ then $\mu(f) = \mu(\Phi(f))$ (see [1, p. 62]). If the characteristic of $\mathbf{K}$ is $p = \operatorname{char}\mathbf{K} > 0$ then we can have $\mu(f) = +\infty$ and $\mu(uf) < +\infty$ for a unit $u \in \mathbf{K}[[x, y]]$ (take $f = x^p + y^{p-1}$ and $u = 1 + x$).

Let $f \in \mathbf{K}[[x, y]]$ be a reduced (without multiple factors) power series and consider a regular parameter $l \in \mathbf{K}[[x, y]]$. Assume that l does not divide f. We call the *polar of* f *with respect to* l the power series

$$\mathscr{P}_l(f) = \frac{\partial(f, l)}{\partial(x, y)} = \frac{\partial f}{\partial x}\frac{\partial l}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial l}{\partial x}.$$

If $l = -bx + ay$ for $(a, b) \neq (0, 0)$ then $\mathscr{P}_l(f) = a\frac{\partial f}{\partial x} + b\frac{\partial f}{\partial y}$.

For any reduced power series f we put $\mathscr{O}_f = \mathbf{K}[[x, y]]/(f)$, $\overline{\mathscr{O}_f}$ the integral closure of $\mathscr{O}_f$ in the total quotient ring of $\mathscr{O}_f$ and $\delta(f) = \dim_{\mathbf{K}} \overline{\mathscr{O}_f}/\mathscr{O}_f$ (the double point number). Let $\mathscr{C}$ be the *conductor* of $\mathscr{O}_f$, that is the largest ideal in $\mathscr{O}_f$ which remains an ideal in $\overline{\mathscr{O}_f}$. We define $c(f) = \dim_{\mathbf{K}} \overline{\mathscr{O}_f}/\mathscr{C}$ (the *degree of conductor*) and $r(f)$ the number of irreducible factors of f. The *semigroup* $\Gamma(f)$ associated with the irreducible power series f is defined as the set of intersection numbers $i_0(f, h)$, where h runs over power series such that $h \not\equiv 0 \pmod f$.

The degree of conductor $c(f)$ is equal to the smallest element c of $\Gamma(f)$ such that $c + N \in \Gamma(f)$ for all integers $N \geq 0$ (see [2, 9]).

For any reduced power series f we define

$$\overline{\mu}(f) := c(f) - r(f) + 1.$$

In particular, if f is irreducible then $\overline{\mu}(f) = c(f)$.

Proposition 2.1 *Let* $f = f_1 \cdots f_r \in \mathbf{K}[[x, y]]$ *be a reduced power series, where* f_i *is irreducible for* $i = 1, \ldots, r$. *Then*

(i) $\overline{\mu}(f) = \overline{\mu}(uf)$ *for any unit* u *of* $\mathbf{K}[[x, y]]$.

(ii)

$$\overline{\mu}(f) + r - 1 = \sum_{i=1}^{r} \overline{\mu}(f_i) + 2 \sum_{1 \leq i < j \leq r} i_0(f_i, f_j).$$

(iii) *Let* l *be a regular parameter such that* $i_0(f_i, l) \not\equiv 0 \pmod p$ *for* $i = 1, \ldots, r$. *Then*

$$i_0(f, \mathscr{P}_l(f)) = \overline{\mu}(f) + i_0(f, l) - 1.$$

(iv) $\overline{\mu}(f) = \mu(f)$ *if and only if* $\mu(f) = 2\delta(f) - r(f) + 1$.

(v) $\overline{\mu}(f) \geq 0$ *and* $\overline{\mu}(f) = 0$ *if and only if* $\operatorname{ord}(f) = 1$.

Proof Property (i) is obvious. To check (ii) observe that

$$\sum_{i=1}^{r}\overline{\mu}(f_i)+2\sum_{1\leq i<j\leq r} i_0(f_i,f_j)=\sum_{i=1}^{r}c(f_i)+2\sum_{1\leq i<j\leq r} i_0(f_i,f_j)=c(f)=\overline{\mu}(f)+r-1,$$

by [3, Lemma 2.1, p. 381]. Property (iii) in the case $r=1$ reduces to the Dedekind formula $i_0(f,\mathscr{P}_l(f))=c(f)+i_0(f,l)-1$ provided that $i_0(f,l)\not\equiv 0 \pmod p$ [7, Lemma 3.1]. To check the general case we apply the Dedekind formula to the irreducible factors f_i of f and we get

$$\begin{aligned}
i_0(f,\mathscr{P}_l(f)) &= \sum_{i=1}^{r} i_0(f_i,\mathscr{P}_l(f)) = \sum_{i=1}^{r} i_0\left(f_i,\mathscr{P}_l(f_i)\frac{f}{f_i}\right)\\
&= \sum_{i=1}^{r}\left(i_0(f_i,\mathscr{P}_l(f_i))+\sum_{j\neq i} i_0(f_i,f_j)\right)\\
&= \sum_{i=1}^{r}\left(\overline{\mu}(f_i)+i_0(f_i,l)-1+\sum_{j\neq i} i_0(f_i,f_j)\right)\\
&= \sum_{i=1}^{r}\overline{\mu}(f_i)+2\sum_{1\leq i<j\leq r} i_0(f_i,f_j)+i_0(f,l)-r\\
&= \overline{\mu}(f)+r-1+i_0(f,l)-r=\overline{\mu}(f)+i_0(f,l)-1.
\end{aligned}$$

Property (iv) follows since $c(f)=2\delta(f)$ for any reduced power series f by the Gorenstein theorem (see for example [20, Section 5]).

Now we prove Property (v). If f is irreducible then $\overline{\mu}(f)=c(f)\geq 0$ with equality if and only if $\mathrm{ord}(f)=1$. Suppose that $r>1$. Then by (ii) we get

$$\overline{\mu}(f)+r-1\geq 2\sum_{1\leq i<j\leq r} i_0(f_i,f_j)\geq r(r-1)$$

and $\overline{\mu}(f)\geq (r-1)^2>0$, which proves (v).

Remark 2.2 Using Proposition 2.1(ii) we check the following property:

Let $f=g_1\cdots g_s\in \mathbf{K}[[x,y]]$ be a reduced power series, where the power series g_i for $i=1,\ldots,s$ are pairwise coprime. Then

$$\overline{\mu}(f)+s-1=\sum_{i=1}^{s}\overline{\mu}(g_i)+2\sum_{1\leq i<j\leq s} i_0(g_i,g_j).$$

Let $\overrightarrow{w} = (n, m) \in (\mathbf{N}_+)^2$ be a pair of strictly positive integers. In the sequel we call $\overrightarrow{w}$ a *weight*.

Let $f = \sum c_{\alpha\beta} x^\alpha y^\beta \in \mathbf{K}[[x, y]]$ be a power series. Then

- the $\overrightarrow{w}$ *-order* of f is $\mathrm{ord}_{\overrightarrow{w}}(f) = \inf\{\alpha n + \beta m \ :\ c_{\alpha\beta} \neq 0\}$,
- the $\overrightarrow{w}$*-initial form* of f is $\mathrm{in}_{\overrightarrow{w}}(f) = \sum_{\alpha n+\beta m=w} c_{\alpha\beta} x^\alpha y^\beta$, where $w = \mathrm{ord}_{\overrightarrow{w}}(f)$,
- $R_{\overrightarrow{w}}(f) = f - \mathrm{in}_{\overrightarrow{w}}(f)$.

Thus $R_{\overrightarrow{w}}(f)$ is a power series of $\overrightarrow{w}$-order greater than $\mathrm{ord}_{\overrightarrow{w}}(f)$.

Note that $\mathrm{ord}_{\overrightarrow{w}}(x) = n$ and $\mathrm{ord}_{\overrightarrow{w}}(y) = m$.

A power series f is *semi-quasihomogeneous* (with respect to $\overrightarrow{w}$) if the system of equations

$$\begin{cases} \frac{\partial}{\partial x}\mathrm{in}_{\overrightarrow{w}}(f) = 0, \\ \\ \frac{\partial}{\partial y}\mathrm{in}_{\overrightarrow{w}}(f) = 0 \end{cases}$$

has the only solution $(x, y) = (0, 0)$.

A power series f is *convenient* if $f(x, 0) \cdot f(0, y) \neq 0$.

Suppose that $\mathrm{in}_{\overrightarrow{w}}(f)$ is convenient and the line $\alpha n + \beta m = \mathrm{ord}_{\overrightarrow{w}}(f)$ intersects the axes in points $(m, 0)$ and $(0, n)$. Let $d = \gcd(m, n)$. Then $\mathrm{in}_{\overrightarrow{w}}(f) = F(x^{m/d}, y^{n/d})$, where $F(u, v) \in \mathbf{K}[u, v]$ is a homogeneous polynomial of degree d.

Proposition 2.3 *Suppose that* $\mathrm{in}_{\overrightarrow{w}}(f)$ *has no multiple factors. Then*

$$\overline{\mu}(f) = \left(\frac{\mathrm{ord}_{\overrightarrow{w}}(f)}{n} - 1\right) \cdot \left(\frac{\mathrm{ord}_{\overrightarrow{w}}(f)}{m} - 1\right).$$

Proof In the proof we will use lemmas collected in the Appendix.

Observe that if $\mathrm{in}_{\overrightarrow{w}}(f)$ has no multiple factors then $\mathrm{in}_{\overrightarrow{w}}(f) = m_{\overrightarrow{w}}(f) \left(\mathrm{in}_{\overrightarrow{w}}(f)\right)^o$, where $m_{\overrightarrow{w}}(f) \in \{1, x, y, xy\}$ and $\left(\mathrm{in}_{\overrightarrow{w}}(f)\right)^o$ is a convenient power series or a constant. To prove the proposition we will use Hensel's Lemma (see Lemma A.3) and Remark 2.2. We have to consider several cases.

Case 1: $\mathrm{in}_{\overrightarrow{w}}(f) = (\mathrm{const}) \cdot x$ or $\mathrm{in}_{\overrightarrow{w}}(f) = (\mathrm{const}) \cdot y$.

In this case $\mathrm{ord}(f) = 1$ and by Proposition 2.1*(v)* $\overline{\mu}(f) = 0$. If $\mathrm{in}_{\overrightarrow{w}}(f) = (\mathrm{const}) \cdot x$ (resp. $\mathrm{in}_{\overrightarrow{w}}(f) = (\mathrm{const}) \cdot y$) then $\mathrm{ord}_{\overrightarrow{w}}(f) = n$ (resp. $\mathrm{ord}_{\overrightarrow{w}}(f) = m$) and

$$\left(\frac{\mathrm{ord}_{\overrightarrow{w}}(f)}{n} - 1\right)\left(\frac{\mathrm{ord}_{\overrightarrow{w}}(f)}{m} - 1\right) = 0.$$

Case 2: $\mathrm{in}_{\overrightarrow{w}}(f) = (\mathrm{const}) \cdot xy$.

By Hensel's Lemma (see Lemma A.3) $f = f_1 f_2$, where $\mathrm{in}_{\overrightarrow{w}}(f_1) = c_1 x$, $\mathrm{in}_{\overrightarrow{w}}(f_2) = c_2 y$ with constants $c_1, c_2 \neq 0$. Using Remark 2.2 and Lemma A.1 we get

$$\overline{\mu}(f) + 1 = \overline{\mu}(f_1 f_2) + 1 = \overline{\mu}(f_1) + \overline{\mu}(f_2) + 2i_0(f_1, f_2) = 0 + 0 + 2.1$$

and $\overline{\mu}(f) = 1$. On the other hand $\mathrm{ord}_{\overrightarrow{w}}(f) = n + m$ and

$$\left(\frac{\mathrm{ord}_{\overrightarrow{w}}(f)}{n} - 1\right)\left(\frac{\mathrm{ord}_{\overrightarrow{w}}(f)}{m} - 1\right) = 1.$$

Case 3: The power series $\mathrm{in}_{\overrightarrow{w}} f$ is convenient.

Assume additionally that the line $n\alpha + m\beta = \mathrm{ord}_{\overrightarrow{w}}(f)$ intersects the axes in points $(m, 0)$ and $(0, n)$. Let $d = \gcd(n, m)$. Then the $\overrightarrow{w}$-initial form of f is

$$\mathrm{in}_{\overrightarrow{w}} f = \prod_{i=1}^{d} \left(a_i x^{m/d} + b_i y^{n/d}\right),$$

where $a_i x^{m/d} + b_i y^{n/d}$ are pairwise coprime. By Hensel's Lemma (see Lemma A.3) we get a factorization $f = \prod_{i=1}^{d} f_i$, where $\mathrm{in}_{\overrightarrow{w}} f_i = a_i x^{m/d} + b_i y^{n/d}$ for $i = 1, \ldots, d$. The factors f_i are irreducible with semigroup $\Gamma(f_i) = \frac{m}{d}\mathbf{N} + \frac{n}{d}\mathbf{N}$ and

$$\overline{\mu}(f_i) = c(f_i) = \left(\frac{m}{d} - 1\right)\left(\frac{n}{d} - 1\right)$$

(see, for example [6]). Moreover by Lemma A.1 we have

$$i_0(f_i, f_j) = \frac{\mathrm{ord}_{\overrightarrow{w}} f_i \,\mathrm{ord}_{\overrightarrow{w}} f_j}{mn} = \frac{mn}{d^2}, \text{ for } i \neq j$$

and we get by Proposition 2.1(ii)

$$\overline{\mu}(f) + d - 1 = \sum_{i=1}^{d} \overline{\mu}(f_i) + 2 \sum_{1 \leq i < j \leq d} i_0(f_i, f_j) = d\left(\frac{m}{d} - 1\right)\left(\frac{n}{d} - 1\right) + 2\frac{d(d-1)}{2}\frac{mn}{d^2},$$

which implies $\overline{\mu}(f) = (m-1)(n-1) = \left(\frac{\mathrm{ord}_{\overrightarrow{w}} f}{n} - 1\right)\left(\frac{\mathrm{ord}_{\overrightarrow{w}} f}{m} - 1\right)$, since the weighted order of f is $\mathrm{ord}_{\overrightarrow{w}} f = mn$.

Now consider the general case, that is when the line $n\alpha + m\beta = \mathrm{ord}_{\overrightarrow{w}}(f)$ intersects the axes in points $(m_1, 0) = \left(\frac{\mathrm{ord}_{\overrightarrow{w}} f}{n}, 0\right)$ and $(0, n_1) = \left(0, \frac{\mathrm{ord}_{\overrightarrow{w}}(f)}{m}\right)$. Then f is semi-quasihomogeneous with respect to $\overrightarrow{w_1} = (n_1, m_1)$ and the line $n_1\alpha + m_1\beta = \mathrm{ord}_{\overrightarrow{w_1}}(f)$ intersects the axes in points $(m_1, 0)$ and $(0, n_1)$. By the first part of the proof we get

$$\overline{\mu}(f) = (m_1 - 1)(n_1 - 1) = \left(\frac{\mathrm{ord}_{\overrightarrow{w}}(f)}{n} - 1\right)\left(\frac{\mathrm{ord}_{\overrightarrow{w}}(f)}{m} - 1\right),$$

which proves the proposition in Case 3.

Case 4: $\mathrm{in}_{\overrightarrow{w}}(f) = x\left(\mathrm{in}_{\overrightarrow{w}}(f)\right)^o$ or $\mathrm{in}_{\overrightarrow{w}}(f) = y\left(\mathrm{in}_{\overrightarrow{w}}(f)\right)^o$, where $\left(\mathrm{in}_{\overrightarrow{w}}(f)\right)^o$ is convenient.

This case follows from Hensel's Lemma (Lemma A.3), Cases 1 and 3.

Case 5: $\mathrm{in}_{\overrightarrow{w}}(f) = xy\left(\mathrm{in}_{\overrightarrow{w}}(f)\right)^o$, where $\left(\mathrm{in}_{\overrightarrow{w}}(f)\right)^o$ is convenient.

This case follows from Hensel's Lemma (Lemma A.3), Cases 2 and 3.

Theorem 2.4 *Suppose that* $\mathrm{in}_{\overrightarrow{w}}(f)$ *has no multiple factors. Then* f *is tame if and only if* f *is a semi-quasihomogeneous singularity with respect to* $\overrightarrow{w}$.

Proof We have $\overline{\mu}(f) = \left(\frac{\mathrm{ord}_{\overrightarrow{w}}(f)}{n} - 1\right)\left(\frac{\mathrm{ord}_{\overrightarrow{w}}(f)}{m} - 1\right)$ by Proposition 2.3. On the other hand, by Lemma A.2, we get that $\mu(f) = \left(\frac{\mathrm{ord}_{\overrightarrow{w}}(f)}{n} - 1\right)\left(\frac{\mathrm{ord}_{\overrightarrow{w}}(f)}{m} - 1\right)$ if and only if the system of equations

$$\begin{cases} \frac{\partial}{\partial x}\mathrm{in}_{\overrightarrow{w}}(f) = 0, \\ \\ \frac{\partial}{\partial y}\mathrm{in}_{\overrightarrow{w}}(f) = 0 \end{cases}$$

has the only solution $(x, y) = (0, 0)$. The theorem follows from Proposition 2.1(iv).

Example 2.5 Let $f(x, y) = x^m + y^n + \sum_{\alpha n+\beta m>nm} c_{\alpha\beta} x^\alpha y^\beta$ and let $d = \gcd(m, n)$. Then $\mathrm{in}_{\overrightarrow{w}}(f) = x^m + y^n$ has no multiple factors if and only if $d \not\equiv 0 \pmod{p}$. If $d \not\equiv 0 \pmod{p}$ then f is tame if and only if $m \not\equiv 0 \pmod{p}$ and $n \not\equiv 0 \pmod{p}$.

Corollary 2.6 *The semi-quasihomogeneous singularities are tame.*

Corollary 2.6 is a particular case of the following

Theorem 2.7 (Boubakri, Greuel, Markwig [1, Theorem 9].) *The planar Newton non-degenerate singularities are tame.*

3 Teissier's Lemma in Characteristic $p \geq 0$

The intersection theoretical approach to the Milnor number in characteristic zero [4] is based on a lemma due to Teissier who proved a more general result (the case of hypersurfaces) in [22, Chapter II, Proposition 1.2]. A general formula on isolated complete intersection singularity is due to Greuel [8] and Lê [14]. In this section we study Teissier's Lemma in arbitrary characteristic $p \geq 0$.

Let $f \in \mathbf{K}[[x, y]]$ be a reduced power series and $l \in \mathbf{K}[[x, y]]$ be a regular parameter. Assume that l does not divide f and consider the polar $\mathscr{P}_l(f) = \frac{\partial f}{\partial x}\frac{\partial l}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial l}{\partial x}$ of f with respect to l. In this section we assume, without loss of generality, that $\mathrm{ord}(l(0, y)) = 1$.

Lemma 3.1 *Let $f \in \mathbf{K}[[x, y]]$ be a reduced power series and $l \in \mathbf{K}[[x, y]]$ be a regular parameter. Then $i_0(l, \mathscr{P}_l(f)) \geq i_0(f, l) - 1$ with equality if and only if $i_0(f, l) \not\equiv 0 \pmod p$.*

Proof Recall that $\mathrm{ord}(l(0, y)) = 1$. Let $\phi(t) = (\phi_1(t), \phi_2(t))$ be a good parametrization of the curve $l(x, y) = 0$ (see [19, Section 2]). In particular $0 = l(\phi(t))$ so $\frac{d}{dt}l(\phi(t)) = 0$. On the other hand we have $\mathrm{ord}(\phi_1(t)) = i_0(x, l) = \mathrm{ord}(l(0, y)) = 1$ and $\phi_1'(0) \neq 0$. Differentiating $f(\phi(t))$ and $l(\phi(t))$ we get

$$\frac{d}{dt}f(\phi(t)) = \frac{\partial f}{\partial x}(\phi(t))\phi_1'(t) + \frac{\partial f}{\partial y}(\phi(t))\phi_2'(t) \tag{1}$$

and

$$0 = \frac{d}{dt}l(\phi(t)) = \frac{\partial l}{\partial x}(\phi(t))\phi_1'(t) + \frac{\partial l}{\partial y}(\phi(t))\phi_2'(t). \tag{2}$$

From (2) we have $\frac{\partial l}{\partial x}(\phi(t))\phi_1'(t) = -\frac{\partial l}{\partial y}(\phi(t))\phi_2'(t)$ and by (1) and the definition of $\mathscr{P}_l(f)$ we get

$$\mathscr{P}_l(f)(\phi(t))\phi_1'(t) = \frac{d}{dt}f(\phi(t))\frac{\partial l}{\partial y}(\phi(t)).$$

Since $\phi_1'(t)$ and $\frac{\partial l}{\partial y}(\phi(t))$ are units in $\mathbf{K}[[t]]$ we have

$$\mathrm{ord}(\mathscr{P}_l(f)(\phi(t))) = \mathrm{ord}\left(\frac{d}{dt}f(\phi(t))\right) \geq \mathrm{ord}(f(\phi(t))) - 1,$$

with equality if and only if $\mathrm{ord}(f(\phi(t))) \not\equiv 0 \pmod p$. Now the lemma follows from the formula $i_0(h, l) = \mathrm{ord}(h(\phi(t)))$ which holds for every power series $h \in \mathbf{K}[[x, y]]$.

Corollary 3.2 *Suppose that $i_0(f, l) = \mathrm{ord}(f) \not\equiv 0 \pmod p$ for a regular parameter $l \in \mathbf{K}[[x, y]]$. Then*

(a) $i_0(l, \mathscr{P}_l(f)) = \mathrm{ord}(f) - 1$,
(b) $\mathrm{ord}(\mathscr{P}_l(f)) = \mathrm{ord}(f) - 1$,
(c) *if h is an irreducible factor of $\mathscr{P}_l(f)$ then $i_0(l, h) = \mathrm{ord}(h)$.*

Proof Property (a) follows immediately from Lemma 3.1. To check (b) observe that we get $\mathrm{ord}(\mathscr{P}_l(f)) = \mathrm{ord}(\mathscr{P}_l(f)) \cdot \mathrm{ord}(l) \leq i_0(l, \mathscr{P}_l(f)) = \mathrm{ord}(f) - 1$, where the last equality follows from (a). The inequality $\mathrm{ord}(\mathscr{P}_l(f)) \geq \mathrm{ord}(f) - 1$ is obvious.

Let $\mathscr{P}_l(f) = \prod_{i=1}^s h_i$, where h_i is irreducible. From (a) and (b) we get

$$0 = i_0(l, \mathscr{P}_l(f)) - \operatorname{ord}(\mathscr{P}_l(f)) = \sum_{i=1}^{s} (i_0(l, h_i) - \operatorname{ord}(h_i)) \, .$$

Since $i_0(l, h_i) \geq \operatorname{ord}(h_i)$ we have $i_0(l, h_i) = \operatorname{ord}(h_i)$ for $i = 1, \ldots, s$ which proves (c).

Proposition 3.3 (Teissier's Lemma in characteristic p.) *Let $f \in \mathbf{K}[[x, y]]$ be a reduced power series. Suppose that*

(i) $i_0(f, l) \not\equiv 0 \pmod p$,
(ii) *for any irreducible factor h of $\mathscr{P}_l(f)$ we get $i_0(l, h) \not\equiv 0 \pmod p$.*

Then

$$i_0(f, \mathscr{P}_l(f)) \leq \mu(f) + i_0(f, l) - 1$$

with equality if and only if

(iii) *for any irreducible factor h of $\mathscr{P}_l(f)$ we get $i_0(f, h) \not\equiv 0 \pmod p$.*

Proof Fix an irreducible factor h of $\mathscr{P}_l(f)$ and let $\psi(t) = (\psi_1(t), \psi_2(t))$ be a good parametrization of the curve $h(x, y) = 0$. Then $\operatorname{ord}(l(\psi(t))) = i_0(l, h) \not\equiv 0 \pmod p$ by (ii) and $\operatorname{ord}\left(\frac{d}{dt} l(\psi(t))\right) = \operatorname{ord}(l(\psi(t))) - 1$. Differentiating $f(\psi(t))$ and $l(\psi(t))$ we get

$$\frac{d}{dt} f(\psi(t)) = \frac{\partial f}{\partial x}(\psi(t))\psi_1'(t) + \frac{\partial f}{\partial y}(\psi(t))\psi_2'(t), \tag{3}$$

and

$$\frac{d}{dt} l(\psi(t)) = \frac{\partial l}{\partial x}(\psi(t))\psi_1'(t) + \frac{\partial l}{\partial y}(\psi(t))\psi_2'(t). \tag{4}$$

Since $\mathscr{P}_l(f)(\psi(t)) = 0$, it follows from (3) and (4) that

$$\frac{d}{dt} f(\psi(t)) \frac{\partial l}{\partial y}(\psi(t)) = \frac{d}{dt} l(\psi(t)) \frac{\partial f}{\partial y}(\psi(t)). \tag{5}$$

Since $\frac{\partial l}{\partial y}(\psi(t))$ is a unit in $\mathbf{K}[[t]]$, taking orders in (5) we have

$$\begin{aligned} \operatorname{ord}(f(\psi(t))) - 1 \leq \operatorname{ord}\left(\frac{d}{dt} f(\psi(t))\right) &= \operatorname{ord}\left(\frac{d}{dt} l(\psi(t))\right) + \operatorname{ord}\left(\frac{\partial f}{\partial y}(\psi(t))\right) \\ &= \operatorname{ord}(l(\psi(t))) - 1 + \operatorname{ord}\left(\frac{\partial f}{\partial y}(\psi(t))\right), \end{aligned}$$

where the last equality follows from $\operatorname{ord}(l(\psi(t))) \not\equiv 0 \pmod{p}$.

Hence $i_0(f, h) \le i_0(l, h) + i_0\left(\frac{\partial f}{\partial y}, h\right)$.

Summing up over all h counted with multiplicities as factors of $\mathscr{P}_l(f)$ we obtain

$$i_0(f, \mathscr{P}_l(f)) \le i_0(l, \mathscr{P}_l(f)) + i_0\left(\frac{\partial f}{\partial y}, \mathscr{P}_l(f)\right). \tag{6}$$

By Lemma 3.1 and assumption (i) we have $i_0\,(l, \mathscr{P}_l(f)) = i_0(f, l) - 1$. Moreover $i_0\left(\frac{\partial f}{\partial y}, \mathscr{P}_l(f)\right) = \mu(f)$ since $\operatorname{ord}(l(0, y)) = 1$ and we get from the equality (6)

$$i_0(f, \mathscr{P}_l(f)) \le \mu(f) + i_0(f, l) - 1.$$

The equality holds if and only if $i_0(f, h) = i_0(l, h) + i_0\left(\frac{\partial f}{\partial y}, h\right)$ for every h, which is equivalent to the condition $i_0(f, h) \not\equiv 0 \pmod{p}$, since $i_0(f, h) \not\equiv 0 \pmod{p}$ if and only if $\operatorname{ord}\left(\frac{d}{dt} f(\psi(t))\right) = \operatorname{ord}(f(\psi(t))) - 1$.

Corollary 3.4 (Teissier [22, Chapter II, Proposition 1.2]) *If* $\operatorname{char} \mathbf{K} = 0$ *then*

$$i_0(f, \mathscr{P}_l(f)) = \mu(f) + i_0(f, l) - 1.$$

Corollary 3.5 *Suppose that* $p = \operatorname{char} \mathbf{K} > \operatorname{ord}(f)$ *and let* $i_0(f, l) = \operatorname{ord}(f)$. *Then*

$$i_0(\mathscr{P}_l(f), f) \le \mu(f) + i_0(f, l) - 1.$$

The equality holds if and only if for any irreducible factor h *of* $\mathscr{P}_l(f)$ *we get* $i_0(f, h) \not\equiv 0 \pmod{p}$.

Proof If $\operatorname{ord}(f) < p$ then $i_0(f, l) = \operatorname{ord}(f) \not\equiv 0 \pmod{p}$ and by Corollary 3.2 for any irreducible factor h of $\mathscr{P}_l(f)$ we get

$$i_0(l, h) = \operatorname{ord}(h) \le \operatorname{ord}(\mathscr{P}_l(f)) = \operatorname{ord}(f) - 1 < p.$$

Hence $i_0(l, h) \not\equiv 0 \pmod{p}$ and the corollary follows from Proposition 3.3.

Example 3.6 Let $f = x^{p+2} + y^{p+1} + x^{p+1}y$, where $p = \operatorname{char} K > 2$. Take $l = y$. Then $i_0(f, l) = p+2 \not\equiv 0 \pmod{p}$, $\mathscr{P}_l(f) = \frac{\partial f}{\partial x} = x^p(2x + y)$ and the irreducible factors of $\mathscr{P}_l(f)$ are $h_1 = x$ and $h_2 = 2x + y$. Clearly $i_0(l, h_1) = i_0(l, h_2) = 1 \not\equiv 0 \pmod{p}$. Moreover $i_0(f, h_1) = i_0(f, h_2) = p + 1$ and all assumptions of Proposition 3.3 are satisfied.

Hence $i_0(f, \mathscr{P}_l(f)) = \mu(f) + i_0(f, l) - 1$ and $\mu(f) = i_0(f, \mathscr{P}_l(f)) - i_0(f, l) + 1 = p(p+1)$. Note that $l = 0$ is a curve of maximal contact with $f = 0$. Let $l_1 = x$. Then $i_0(f, l_1) = \operatorname{ord}(f) = p+1$, $\mathscr{P}_{l_1}(f) = -(y^p + x^{p+1})$ and $h = y^p + x^{p+1}$ is the only irreducible factor of the polar $\mathscr{P}_{l_1}(f)$. Since $i_0(l_1, h) = p$, the condition (ii) of Proposition 3.3 is not satisfied. However, $i_0(f, \mathscr{P}_{l_1}(f)) = \mu(f) + i_0(f, l_1) - 1$, which we check directly.

4 Tame Singularities

Assume that f is a plane curve singularity.

Proposition 4.1 *Let $f = f_1 \cdots f_r \in \mathbf{K}[[x, y]]$ be a reduced power series, where f_i is irreducible for $i = 1, \ldots, r$. Suppose that there exists a regular parameter l such that $i_0(f_i, l) \not\equiv 0 \pmod{p}$ for $i = 1, \ldots, r$. Then f is tame if and only if Teissier's lemma holds, that is if $i_0(f, \mathscr{P}_l(f)) = \mu(f) + i_0(f, l) - 1$.*

Proof By Proposition 2.1(iii) we have that $i_0(f, \mathscr{P}_l(f)) = \overline{\mu}(f) + i_0(f, l) - 1$. Thus $i_0(f, \mathscr{P}_l(f)) = \mu(f) + i_0(f, l) - 1$ if and only if $\mu(f) = \overline{\mu}(f)$. We finish the proof using Proposition 2.1(iv).

Proposition 4.2 (Milnor [17], Risler [21]) *If $\operatorname{char} \mathbf{K} = 0$ then any plane singularity is tame.*

Proof Teissier's Lemma holds by Corollary 3.4. Use Proposition 4.1.

Proposition 4.3 *Let $p = \operatorname{char} \mathbf{K} > 0$. Suppose that $p > \operatorname{ord}(f)$. Let l be a regular parameter such that $i_0(f, l) = \operatorname{ord}(f)$. Then f is tame if and only if for any irreducible factor h of $\mathscr{P}_l(f)$ we get $i_0(f, h) \not\equiv 0 \pmod{p}$.*

Proof Take a regular parameter l such that $i_0(f, l) = \operatorname{ord}(f)$. By hypothesis we get $i_0(f, l) < p$ so $i_0(f, l) \not\equiv 0 \pmod{p}$. By Corollary 3.2 the assumption (ii) of Proposition 3.3 is satisfied.

Hence $i_0(f, \mathscr{P}_l(f)) \leq \mu(f) + i_0(f, l) - 1$ with equality if and only if $i_0(f, h) \not\equiv 0 \pmod{p}$ for any irreducible factor h of $\mathscr{P}_l(f)$. Use Proposition 4.1.

Proposition 4.4 (Nguyen [18]) *Let $p = \operatorname{char} \mathbf{K} > 0$. Suppose that there exists a regular parameter l such that $i_0(f, l) = \operatorname{ord}(f)$ and $i_0(f, \mathscr{P}_l(f)) < p$. Then f is tame.*

Proof We have $p > i_0(f, \mathscr{P}_l(f)) \geq \operatorname{ord}(f) \cdot \operatorname{ord}(\mathscr{P}_l(f))$. Hence $p > \operatorname{ord}(f)$ and we may apply Proposition 4.3. Since $i_0(f, \mathscr{P}_l(f)) < p$ for any irreducible factor h of $\mathscr{P}_l(f)$ we have that $i_0(f, h) < p$ and obviously $i_0(f, h) \not\equiv 0 \pmod{p}$. The proposition follows from Proposition 4.3.

Theorem 4.5 (Nguyen [18]) *If $p > \mu(f) + \operatorname{ord}(f) - 1$ then f is tame.*

Proof Since f is a singularity we get $\mu(f) > 0$ and by hypothesis the characteristic of the field verifies $p > \mu(f) - 1 + \operatorname{ord}(f) \geq \operatorname{ord}(f)$. By the first part of the proof of Proposition 4.3 we have $i_0(f, \mathscr{P}_l(f)) \leq \mu(f) + \operatorname{ord}(f) - 1$, where l is a regular parameter such that $i_0(f, l) = \operatorname{ord}(f)$. Hence $i_0(f, \mathscr{P}_l(f)) < p$ and the theorem follows from Proposition 4.4.

5 The Milnor Number of Plane Irreducible Singularities

Let $f \in \mathbf{K}[[x, y]]$ be an irreducible power series of order $n = \mathrm{ord}(f)$ and let $\Gamma(f)$ be the semigroup associated with $f = 0$.

Let $\overline{\beta_0}, \dots, \overline{\beta_g}$ be the minimal sequence of generators of $\Gamma(f)$ defined by the conditions

- $\overline{\beta_0} = \min(\Gamma(f)\backslash\{0\}) = \mathrm{ord}(f) = n$,
- $\overline{\beta_k} = \min(\Gamma(f)\backslash\mathbf{N}\overline{\beta_0} + \cdots + \mathbf{N}\overline{\beta_{k-1}})$ for $k \in \{1, \dots, g\}$,
- $\Gamma(f) = \mathbf{N}\overline{\beta_0} + \cdots + \mathbf{N}\overline{\beta_g}$.

Let $e_k = \gcd(\overline{\beta_0}, \dots, \overline{\beta_k})$ for $k \in \{1, \dots, g\}$. Then $n = e_0 > e_1 > \cdots e_{g-1} > e_g = 1$. Let $n_k = e_{k-1}/e_k$ for $k \in \{1, \dots, g\}$. We have $n_k > 1$ for $k \in \{1, \dots, g\}$ and $n = n_1 \cdots n_g$. Let $n^* = \max(n_1, \dots, n_g)$. Then $n^* \leq n$ with equality if and only if $g = 1$.

The following theorem is a sharpened version of the main result of [7].

Theorem 5.1 *Let $f \in \mathbf{K}[[x, y]]$ be an irreducible power series of order $n > 1$ and let $\overline{\beta_0}, \dots, \overline{\beta_g}$ be the minimal system of generators of $\Gamma(f)$. Suppose that $p = \mathrm{char}\,\mathbf{K} > n^*$. Then the following two conditions are equivalent:*

(i) $\overline{\beta_k} \not\equiv 0 \pmod p$ *for* $k \in \{1, \dots, g\}$,
(ii) f *is tame.*

In [7] the equivalence of (i) and (ii) is proved under the assumption that $p > n$.

If $f \in \mathbf{K}[[x, y]]$ is an irreducible power series then we get $\mathrm{ord}(f(x, 0)) = \mathrm{ord}(f)$ or $\mathrm{ord}(f(0, y)) = \mathrm{ord}(f)$. In the sequel we assume that $\mathrm{ord}(f(0, y)) = \mathrm{ord}(f) = n$. The proof of Theorem 5.1 is based on Merle's factorization theorem:

Theorem 5.2 (Merle [15], García Barroso-Płoski [7]) *Suppose that* $\mathrm{ord}(f(0, y)) = \mathrm{ord}(f) = n \not\equiv 0 \pmod p$. *Then* $\frac{\partial f}{\partial y} = h_1 \cdots h_g$ *in* $\mathbf{K}[[x, y]]$, *where*

(a) $\mathrm{ord}(h_k) = \frac{n}{e_k} - \frac{n}{e_{k-1}}$ *for* $k \in \{1, \dots, g\}$.
(b) *If* $h \in \mathbf{K}[[x, y]]$ *is an irreducible factor of* h_k, $k \in \{1, \dots, g\}$, *then*

  (b1) $\frac{i_0(f,h)}{\mathrm{ord}(h)} = \frac{e_{k-1}\overline{\beta_k}}{n}$, *and*
  (b2) $\mathrm{ord}(h) \equiv 0 \left(\mathit{mod}\ \frac{n}{e_{k-1}}\right)$.

Lemma 5.3 *Suppose that $p > n^*$. Then $i_0\left(f, \frac{\partial f}{\partial y}\right) \leq \mu(f) + \mathrm{ord}(f) - 1$ with equality if and only if $\overline{\beta_k} \not\equiv 0 \pmod p$ for $k \in \{0, \dots, g\}$.*

Proof Obviously $n_k \not\equiv 0 \pmod p$ for $k = 1, \dots, g$ and $n = n_1 \cdots n_g \not\equiv 0 \pmod p$. Let h be an irreducible factor of $\frac{\partial f}{\partial y}$. Then, by Corollary 3.2(c) $i_0(h, x) = \mathrm{ord}(h)$. By Theorem 5.2 (b2) $\mathrm{ord}(h) = m_k \frac{n}{e_{k-1}}$, for an index $k \in \{1, \dots, g\}$, where $m_k \geq 1$ is an integer. Hence $m_k \frac{n}{e_{k-1}} = \mathrm{ord}(h) \leq \mathrm{ord}(h_k) = \frac{n}{e_{k-1}}(n_k - 1)$ and $m_k \leq n_k - 1 < n_k < p$, which implies $m_k \not\equiv 0 \pmod p$ and $\mathrm{ord}(h) \not\equiv 0 \pmod p$. By Proposition 3.3 we get $i_0\left(f, \frac{\partial f}{\partial y}\right) \leq \mu(f) + \mathrm{ord}(f) - 1$. By Theorem 5.2 (b1) we have the equalities $i_0(f,h) = \left(\frac{e_{k-1}\overline{\beta_k}}{n}\right)\mathrm{ord}(h) = m_k\overline{\beta_k}$ and we get $i_0(f,h) \not\equiv 0 \pmod{p}$ if and only if $\overline{\beta_k} \not\equiv 0 \pmod{p}$, which proves the second part of Lemma 5.3.

Proof of Theorem 5.1 Use Lemma 5.3 and Proposition 4.1. ■

Example 5.4 Let $f(x,y) = (y^2 + x^3)^2 + x^5 y$. Then f is irreducible and its semigroup is $\Gamma(f) = 4\mathbf{N} + 6\mathbf{N} + 13\mathbf{N}$. Here $e_0 = 4$, $e_1 = 2$, $e_2 = 1$ and $n_1 = n_2 = 2$. Hence $n^* = 2$.

Let $p > n^* = 2$. If $p = \mathrm{char}\,\mathbf{K} \neq 3, 13$ then f is tame. On the other hand if $p = 2$ then $\mu(f) = +\infty$ since x is a common factor of $\frac{\partial f}{\partial y}$ and $\frac{\partial f}{\partial x}$. Hence f is tame if and only if $p \neq 2, 3, 13$. Note that for any l with $\mathrm{ord}(l) = 1$ we have $i_0(f,l) \equiv 0 \pmod{2}$.

Proposition 5.5 *If* $\Gamma(f) = \overline{\beta_0}\mathbf{N} + \overline{\beta_1}\mathbf{N}$ *then f is tame if and only if* $\overline{\beta_0} \not\equiv 0 \pmod{p}$ *and* $\overline{\beta_1} \not\equiv 0 \pmod{p}$.

Proof Let $\overrightarrow{w} = (\overline{\beta_0}, \overline{\beta_1})$. There exists a system of coordinates x, y such that we can write $f = y^{\overline{\beta_0}} + x^{\overline{\beta_1}} +$ terms of weight greater than $\overline{\beta_0}\,\overline{\beta_1}$. The proposition follows from Theorem 2.4 (see also [7, Example 2]).

In [11] the authors proved, without any restriction on $p = \mathrm{char}\,\mathbf{K}$, the following profound result:

Theorem 5.6 (Hefez, Rodrigues, Salomão [11, 12]) *Let* $\Gamma(f) = \overline{\beta_0}\mathbf{N} + \cdots + \overline{\beta_g}\mathbf{N}$. *If* $\overline{\beta_k} \not\equiv 0 \pmod{p}$ *for* $k = 0, \ldots, g$ *then f is tame.*

The question as to whether the converse of Theorem 5.6 is true remains open.

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Appendix

Let $\overrightarrow{w} = (n, m) \in (\mathbf{N}_+)^2$ be a weight.

Lemma A.1 *Let* $f, g \in \mathbf{K}[[x,y]]$ *be power series without constant term. Then*

$$i_0(f,g) \geq \frac{\left(\mathrm{ord}_{\overrightarrow{w}}(f)\right)\left(ord_{\overrightarrow{w}}(g)\right)}{mn},$$

with equality if and only if the system of equations

$$\begin{cases} \mathrm{in}_{\overrightarrow{w}}(f) = 0, \\ \\ \mathrm{in}_{\overrightarrow{w}}(g) = 0 \end{cases}$$

has the only solution $(x, y) = (0, 0)$.

Proof By a basic property of the intersection multiplicity (see for example [19, Proposition 3.8 (v)]) we have that for any nonzero power series $\tilde{f}, \tilde{g}$

$$i_0(\tilde{f}, \tilde{g}) \geq \mathrm{ord}(\tilde{f})\mathrm{ord}(\tilde{g}), \tag{7}$$

with equality if and only if the system of equations $\mathrm{in}(\tilde{f}) = 0$, $\mathrm{in}(\tilde{g}) = 0$ has the only solution $(0, 0)$. Consider the power series $\tilde{f}(u, v) = f(u^n, v^m)$ and $\tilde{g}(u, v) = g(u^n, v^m)$. Then $i_0(\tilde{f}, \tilde{g}) = i_0(f, g)i_0(u^n, v^m) = i_0(f, g)nm$, $\mathrm{ord}(\tilde{f}) = \mathrm{ord}_{\overrightarrow{w}}(f)$, $\mathrm{ord}(\tilde{g}) = \mathrm{ord}_{\overrightarrow{w}}(g)$ and the lemma follows from (7).

Lemma A.2 *Let* $f \in \mathbf{K}[[x, y]]$ *be a non-zero power series. Then*

$$i_0\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) \geq \left(\frac{\mathrm{ord}_{\overrightarrow{w}}(f)}{n} - 1\right)\left(\frac{\mathrm{ord}_{\overrightarrow{w}}(f)}{m} - 1\right)$$

with equality if and only if f *is a semi-quasihomogeneous singularity with respect to* $\overrightarrow{w}$.

Proof The following two properties are useful:

$$\mathrm{ord}_{\overrightarrow{w}}\left(\frac{\partial f}{\partial x}\right) \geq \mathrm{ord}_{\overrightarrow{w}}(f) - n \quad \text{with equality if and only if} \quad \frac{\partial}{\partial x}\mathrm{in}_{\overrightarrow{w}}(f) \neq 0, \tag{8}$$

$$\text{if } \frac{\partial}{\partial x}\mathrm{in}_{\overrightarrow{w}}(f) \neq 0 \text{ then } \mathrm{in}_{\overrightarrow{w}}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\mathrm{in}_{\overrightarrow{w}}(f). \tag{9}$$

By the first part of Lemma A.1 and Property (8) we get

$$\begin{aligned} i_0\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) &\geq \frac{\left(\mathrm{ord}_{\overrightarrow{w}}\left(\frac{\partial f}{\partial x}\right)\right)\left(ord_{\overrightarrow{w}}\left(\frac{\partial f}{\partial y}\right)\right)}{nm} \geq \frac{\left(\mathrm{ord}_{\overrightarrow{w}}(f) - n\right)\left(ord_{\overrightarrow{w}}(f) - m\right)}{nm} \\ &= \left(\frac{\mathrm{ord}_{\overrightarrow{w}}(f)}{n} - 1\right)\left(\frac{\mathrm{ord}_{\overrightarrow{w}}(f)}{m} - 1\right). \end{aligned}$$

Using the second part of Lemma A.1 and Properties (8) and (9) we check that $i_0\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = \left(\frac{\mathrm{ord}_{\overrightarrow{w}}(f)}{n} - 1\right)\left(\frac{\mathrm{ord}_{\overrightarrow{w}}(f)}{m} - 1\right)$ if and only if f is a semi-quasihomogeneous singularity with respect to $\overrightarrow{w}$.

Lemma A.3 (Hensel's Lemma [13, Theorem 16.6]) *Suppose that* $\mathrm{in}_{\overrightarrow{w}}(f) = \psi_1 \cdots \psi_s$ *with pairwise coprime* ψ_i*. Then* $f = g_1 \cdots g_s \in \mathbf{K}[[x, y]]$ *with* $\mathrm{in}_{\overrightarrow{w}}(g_i) = \psi_i$ *for* $i = 1, \ldots, s$*.*

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