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ON THE INTERSECTION MULTIPLICITY OF PLANE BRANCHES

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Abstract. We prove an intersection formula for two plane branches in terms of their semigroups and key polynomials. Then we provide a strong version of Bayer's theorem on the set of intersection multiplicities of two branches with fixed characteristics and apply it to the logarithmic distance in the space of branches.

1. Introduction. The traditional approach to the study of the singularities of irreducible plane algebroid curves (branches) defined over algebraically closed fields of arbitrary characteristic is based on the Hamburger–Noether expansions which encode the sequences of quadratic transformations appearing in the desingularization of a curve (see [An], [C], [D1], [R]).

In [GB-P] we developed a new approach to the theory of plane branches. We used the logarithmic distance on the set of branches without resorting to the Hamburger–Noether expansions or to the resolution process. This note is written in the spirit of [GB-P]. The concepts of semigroup associated with a branch and key polynomials explained in [GB-P] play an important role.

Let f, g be irreducible power series in $\mathbf{K}[[x, y]]$, where $\mathbf{K}$ is an algebraically closed field. The intersection multiplicity $i_0(f, g)$ of branches $\{f = 0\}$ and $\{g = 0\}$ is a basic notion of the local geometry of plane algebraic curves. The classical formula for $i_0(f, g)$ allows one to calculate the intersection multiplicity in terms of Puiseux parametrizations (see [vdW], [He]) when $\operatorname{char} \mathbf{K} = 0$. If $\mathbf{K}$ has a positive characteristic, a similar result can be obtained by using the Hamburger–Noether expansions (see [An], [C], [R]). The aim of this note is to prove a formula for $i_0(f, g)$ in terms of semigroups $\Gamma(f)$ and $\Gamma(g)$ associated with f and g and key polynomials f_i and g_j which define (in generic coordinates) the maximal contact curves $\{f_i = 0\}$ and $\{g_j = 0\}$. We impose no condition on the characteristic of $\mathbf{K}$.

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In Section 2 we recall the main properties of the semigroup and key polynomials associated with a branch. In Sections 3 and 4 we present the main result (Theorem 3.1) and its proof. Then we give in Section 5 an application of the main result to polynomial automorphisms of the affine plane (Theorem 5.2). In Section 6 we prove a strong version (due to Hefez in characteristic 0) of Bayer's theorem on the set of intersection multiplicities of two branches with fixed characteristics (Theorem 6.1). Section 7 is devoted to a short proof of a property of the logarithmic distance

$$d(f,g) = \frac{i_0(f,g)}{\operatorname{ord} f \operatorname{ord} g}$$

(Theorem 7.1) discovered by Abío et al. [Ab-Al-G] in the case of characteristic 0.

2. Preliminaries. In this note we use the basic notions and theorems of the theory of plane branches explained in [GB-P].

Let $\mathbf{K}$ be an algebraically closed field of arbitrary characteristic. For any power series $f, g \in \mathbf{K}[[x,y]]$ we define the *intersection multiplicity* $i_0(f,g)$ by putting

$$i_0(f,g) = \dim_{\mathbf{K}} \mathbf{K}[[x,y]]/(f,g),$$

where (f,g) is the ideal of $\mathbf{K}[[x,y]]$ generated by f and g. If f, g are non-zero power series without constant term then $i_0(f,g) < \infty$ if and only if f and g are coprime.

Let $f \in \mathbf{K}[[x,y]]$ be an irreducible power series. By definition, the *branch* $\{f=0\}$ is the ideal generated by f in $\mathbf{K}[[x,y]]$. The *multiplicity* of $\{f=0\}$ is the order of the power series f.

For any branch $\{f=0\}$ we put

$$\Gamma(f) = \{i_0(f,g) : g \text{ runs over all power series such that } g \not\equiv 0 \pmod{f}\}.$$

Then $\Gamma(f)$ is a semigroup. We call $\Gamma(f)$ the *semigroup associated with the branch* $\{f=0\}$.

Two branches $\{f=0\}$ and $\{g=0\}$ are *equisingular* if $\Gamma(f) = \Gamma(g)$. The branch $\{f=0\}$ is *non-singular* (that is, of multiplicity 1) if and only if $\Gamma(f) = \mathbb{N}$. We have $\min(\Gamma(f) \setminus \{0\}) = \operatorname{ord} f$.

Let $n > 0$ be an integer. A sequence $(v_0, \ldots, v_h)$ of positive integers is said to be an *n-characteristic sequence* if $v_0 = n$ and if the following two conditions are fulfilled:

(char 1) Let $e_k = \gcd(v_0, \ldots, v_k)$ for $0 \le k \le h$. Then $n = e_0 > e_1 > \cdots > e_h = 1$.

(char 2) $e_{k-1}v_k < e_k v_{k+1}$ for $1 \le k \le h-1$.

Let $n_k = e_{k-1}/e_k$ for $1 \le k \le h$.

Conditions (char 1) and (char 2) imply *Bézout's relation*:

$$n_k v_k = a_0 v_0 + a_1 v_1 + \cdots + a_{k-1} v_{k-1},$$

where $a_0 > 0$, $0 \le a_i < n_i$ for $1 \le i \le k$ are integers.

Let $f = f(x,y)$ be an irreducible power series. Suppose that $n = i_0(f,x) = \operatorname{ord} f(0,y) < \infty$. Then the *$n$-minimal system* $(v_0,\dots,v_h)$ *of generators* of $\Gamma(f)$ defined by the conditions:

(gen 1) $v_0 = n$, v_k is the smallest element of $\Gamma(f)$ which does not belong to $v_0\mathbb{N} + \cdots + v_{k-1}\mathbb{N}$,

(gen 2) $v_0\mathbb{N} + \cdots + v_h\mathbb{N} = \Gamma(f)$,

is an n-characteristic sequence. We will call it the *characteristic* of $\{f=0\}$ and write $\overline{\mathrm{char}}_x f = (v_0,\dots,v_h)$. If $(v_0,\dots,v_h)$ is the n-minimal system of generators of $\Gamma(f)$ then the number $c = \sum_{k=1}^{h}(n_k-1)v_k - v_0 + 1$ is the *conductor* of $\Gamma(f)$, that is, $c + N \in \Gamma(f)$ for $N \ge 0$ and $c-1 \notin \Gamma(f)$.

There exists a sequence of monic polynomials $f_0,\dots,f_{h-1} \in \mathbf{K}[[x]][y]$ such that $\deg_y f_k = n/e_k$ and $i_0(f,f_k) = v_{k+1}$ for $k = 0,\dots,h-1$.

Recall that a monic polynomial $p \in \mathbf{K}[[x]][y]$ of y-degree $n > 0$ is *distinguished* if $\operatorname{ord} p(0,y) = n$. By the Weierstrass Preparation Theorem there is a unique distinguished polynomial $f_h \in \mathbf{K}[[x]][y]$ associated with f. Then $\deg_y f_h = n = n/e_h$ and $i_0(f,f_h) = \infty$. We put $v_{h+1} = \infty$.

The polynomials $f_0,\dots,f_h \in \mathbf{K}[[x]][y]$ are called *key polynomials* of f. They are not uniquely determined by f. Recall that for any n-characteristic sequence $(v_0,\dots,v_h)$ there exists an irreducible series f such that $\Gamma(f) = v_0\mathbb{N} + \cdots + v_h\mathbb{N}$ and $i_0(f,x) = v_0$ (see [GB-P, Theorem 6.5]).

The basic properties of key polynomials are:

(key 1) A key polynomial f_k is a distinguished, irreducible polynomial of characteristic $\overline{\mathrm{char}}_x f_k = (v_0/e_k,\dots,v_k/e_k)$, hence $\deg_y f_k = v_0/e_k$.

(key 2) Let $(v_0,\dots,v_h)$ be an n-characteristic sequence. Let f_k be an irreducible distinguished polynomial with $\overline{\mathrm{char}}_x f_k = (v_0/e_k,\dots,v_k/e_k)$. Let $f_0, f_1,\dots,f_{k-1}$ be a sequence of key polynomials of f_k. Put $f_i = f_{i-1}^{n_i} + \xi_i x^{a_{i,0}} f_0^{a_{i,1}} \cdots f_{i-2}^{a_{i,i-1}}$ for $i \in \{k+1,\dots,h\}$, where $n_i v_i = a_{i,0}v_0 + \cdots + a_{i,i-1}v_{i-1}$ is a Bézout relation and $\xi_i \in \mathbf{K}\setminus\{0\}$. Then $\overline{\mathrm{char}}_x f_h = (v_0,\dots,v_h)$ and $f_0,\dots,f_{h-1}$ are key polynomials of f_h.

(key 3) If $g \in \mathbf{K}[[x]][y]$ is a monic polynomial such that $\deg_y g = n/e_k$ then $i_0(f,g) \le i_0(f,f_k) = v_{k+1}$.

(key 4) Let $\{g=0\} \ne \{x=0\}$ be a branch of characteristic $(v'_0,\dots,v'_{h'})$. Let $v_0 = i_0(f,x) > 1$ and suppose $i_0(f,g)/i_0(x,g) > i_0(f,f_{k-1})/i_0(x,f_{k-1})$ for some $k \in \{1,\dots,h\}$. Then $k \le h'$ and $v_i/v_0 = v'_i/v'_0$ for all $i \in \{1,\dots,k\}$. The first k key polynomials $f_0,\dots,f_{k-1}$ of f are the first k elements of a sequence of key polynomials of g.

For the proofs of (key 1) and (key 2) we refer the reader to [GB-P, Proposition 4.2 and Theorem 6.1]. For the proof of (key 3) see [GB-P, Lemma 3.12]. For the proof of (key 4) see [GB-P, Theorem 5.2].

The following theorem is a local version of the Abhyankar–Moh result:

ABHYANKAR–MOH IRREDUCIBILITY CRITERION. *Let $f(x,y) \in \mathbf{K}[[x,y]]$ be an irreducible power series such that $n = i_0(f,x) < \infty$ and let $\Gamma(f) = v_0\mathbb{N}+\cdots+v_h\mathbb{N}$, where $v_0 = n$. If $g(x,y) \in \mathbf{K}[[x,y]]$ is a power series such that $i_0(g,x) = n$ and $i_0(f,g) > e_{h-1}v_h$ then g is irreducible and $\Gamma(g) = \Gamma(f)$.*

The proof of the above criterion is given in [GB-P, Corollary 5.8].

3. Main result. Let $\{f = 0\}$ and $\{g = 0\}$ be two branches different from $\{x = 0\}$. Let $\overline{\mathrm{char}}_x f = (v_0,\dots,v_h)$, where $v_0 = n = i_0(f,x)$ and $\overline{\mathrm{char}}_x g = (v'_0,\dots,v'_{h'})$, where $v'_0 = n' = i_0(g,x)$. We denote by $f_0, f_1,\dots,f_h$ and $g_0, g_1,\dots,g_{h'}$ a sequence of key polynomials of f and g, respectively.

THEOREM 3.1 (Intersection formula). *With the assumptions and notations introduced above, there is an integer $0 < k \le \min\{h,h'\}+1$ such that*

(a) *$v_i/n = v'_i/n'$ for all $i < k$.*
(b) *$i_0(f,g) \le \inf\{e'_{k-1}v_k, e_{k-1}v'_k\}$.*
(c) *If $i_0(f,g) < \inf\{e'_{k-1}v_k, e_{k-1}v'_k\}$ then $i_0(f,g) = e_{k-1}e'_{k-1}i_0(f_{k-1},g_{k-1})$.*
(d) *Suppose that $k > 1$. Then $i_0(f,g) > \inf\{e'_{k-2}v_{k-1}, e_{k-2}v'_{k-1}\}$.*

Moreover $f_0,\dots,f_{k-2}$ are the first $k-1$ polynomials of a sequence of key polynomials of g and $g_0,\dots,g_{k-2}$ are the first $k-1$ polynomials of a sequence of key polynomials of f.

REMARK 3.2. From the first part of Theorem 3.1 it follows that $n/e_i = n'/e'_i$ for $i < k$. In fact,

$$\begin{aligned} ne'_i &= n\gcd(v'_0,\dots,v'_i) = \gcd(nv'_0,\dots,nv'_i) = \gcd(n'v_0,\dots,n'v_i) \\ &= n'\gcd(v_0,\dots,v_i) = n'e_i. \end{aligned}$$

Consequently, $n_i = n'_i$ and $e'_{i-1}v_i = e_{i-1}v'_i$ for $0 < i < k$.

The proof of Theorem 3.1 is given in Section 4. Observe that the integer $k > 0$ is the smallest integer such that condition (b) of Theorem 3.1 holds.

COROLLARY 3.3 (see [D2, Lemma 1.7]). *Let $k > 0$ be the minimum integer such that*

$$i_0(f,g) \le \inf\{e'_{k-1}v_k, e_{k-1}v'_k\}.$$

Then

(1) *$v_i/n = v'_i/n'$ for all $i < k$.*
(2) *If $i_0(f,g) < \inf\{e'_{k-1}v_k, e_{k-1}v'_k\}$ then $i_0(f,g) \equiv 0 \pmod{e_{k-1}e'_{k-1}}$.*

F. Delgado [D1, Section 3], [D2, pp. 335–336] computed the integer $k > 0$ in terms of the Hamburger–Noether expansions of f and g. In what follows we do not need any additional information about k.

4. Proof of Theorem 3.1. For any branches $\{f = 0\}$ and $\{g = 0\}$ different from the branch $\{x = 0\}$ we put

$$d_x(f,g) = \frac{i_0(f,g)}{i_0(f,x)i_0(g,x)}.$$

The function d_x satisfies the *Strong Triangle Inequality* (*STI*): for any branches $\{f = 0\}$, $\{g = 0\}$ and $\{h = 0\}$ different from $\{x = 0\}$,

$$d_x(f,g) \geq \inf\{d_x(f,h), d_x(g,h)\},$$

which is equivalent to: *at least two of the numbers $d_x(f,g)$, $d_x(f,h)$, $d_x(g,h)$ are equal and the third is no smaller than the other two* (see [GB-P, Theorem 2.8]).

LEMMA 4.1. *If $n = 1$ then $i_0(f,g) = \inf\{e_0' i_0(f_0, g_0), v_1'\}$.*

Proof. If $n = 1$ then $e_0 = 1$, $h = 0$ and the only possible value for k, $0 < k \leq \min(h+1, h'+1)$ is $k = 1$. Note that $d_x(f,g) = i_0(f,g)/v_0'$, $d_x(f,g_0) = i_0(f,g_0)$, $d_x(g,g_0) = v_1'/v_0' \notin \mathbb{N}$ (if $n' = 1$ then $v_1' = +\infty$ and $d_x(g,g_0) = +\infty$). Therefore $d_x(f,g_0) \neq d_x(g,g_0)$ and $d_x(f,g) = \inf\{d_x(f,g_0), d_x(g,g_0)\}$, which is equivalent to $i_0(f,g) = \inf\{e_0' i_0(f,g_0), v_1'\} = \inf\{e_0' i_0(f_0,g_0), v_1'\}$, since f_0 is the distinguished polynomial associated with f. ∎

Lemma 4.1 implies Theorem 3.1 for $n = 1$. So now we assume $n > 1$.

Let $f_0, \ldots, f_h$ be a sequence of key polynomials of f. Then $d_x(f, f_{k-1}) = e_{k-1}v_k/n^2$. So the sequence $d_x(f, f_{k-1})$ for $k = 1, \ldots, h$ is strictly increasing.

LEMMA 4.2. *Suppose $d_x(f,g) > d_x(f, f_{k-1})$ for an integer $k \in \{1, \ldots, h\}$. Then $k \leq h'$, $d_x(f, f_{i-1}) = d_x(g, g_{i-1})$ for $i = 1, \ldots, k$, and $f_0, \ldots, f_{k-1}$ are the first $k-1$ polynomials of a sequence of key polynomials of g.*

Proof. The lemma follows directly from (key 4). ∎

LEMMA 4.3. *Let $k \leq \min\{h, h'\} + 1$. Suppose $d_x(f, f_{i-1}) = d_x(g, g_{i-1})$ for $0 < i < k$ and $d_x(f, f_{k-1}) \neq d_x(g, g_{k-1})$. Then*

$$d_x(f,g) \leq \inf\{d_x(f, f_{k-1}), d_x(g, g_{k-1})\}.$$

Proof. From $d_x(f, f_{i-1}) = d_x(g, g_{i-1})$ for $0 < i < k$ we get $v_i/n = v_i'/n'$ and $e_i/n = e_i'/n'$ for $0 < i < k$. Thus $\deg_y f_{k-1} = n/e_{k-1} = n'/e_{k-1}' = \deg_y g_{k-1}$. We may assume that $d_x(g, g_{k-1}) < d_x(f, f_{k-1})$. Since $\deg_y f_{k-1} = \deg_y g_{k-1}$, we deduce, applying (key 3) to g, that $i_0(g, f_{k-1}) \leq i_0(g, g_{k-1})$ and consequently $d_x(g, f_{k-1}) \leq d_x(g, g_{k-1})$. Thus $d_x(g, f_{k-1}) < d_x(f, f_{k-1})$, and by the STI, $d_x(g, f_{k-1}) = d_x(f,g)$. Therefore $d_x(f,g) = d_x(g, f_{k-1}) \leq d_x(g, g_{k-1}) = \inf\{d_x(f, f_{k-1}), d_x(g, g_{k-1})\}$. ∎

LEMMA 4.4. *Let* $0 < k \le h+1$ *be the smallest integer such that* $d_x(f,g) \le d_x(f,f_{k-1})$. *Then* $k \le h'+1$ *and* $d_x(f,g) \le d_x(g,g_{k-1})$.

Proof. First we suppose that $k = 1$. Then the inequality $k \le h'+1$ is obvious and the assertion follows from Lemma 4.3.

Suppose now that $k > 1$. By definition of k we have $d_x(f,g) > d_x(f,f_{k-2})$. Then by Lemma 4.2, $k \le h'+1$ and $d_x(f,f_{i-1}) = d_x(g,g_{i-1})$ for $i = 1,\dots,k-1$. If $d_x(f,f_{k-1}) = d_x(g,g_{k-1})$ then the lemma is obvious. If $d_x(f,f_{k-1}) \ne d_x(g,g_{k-1})$ then we use Lemma 4.3. ∎

Proof of Theorem 3.1. Recall that $n > 1$. The assertions of the theorem can be rewritten in the following form:

(a′) If $k > 1$ then $d_x(f,f_{i-1}) = d_x(g,g_{i-1})$ for all $0 < i < k$.
(b′) $d_x(f,g) \le \inf\{d_x(f,f_{k-1}), d_x(g,g_{k-1})\}$.
(c′) If $d_x(f,g) < \inf\{d_x(f,f_{k-1}), d_x(g,g_{k-1})\}$ then $d_x(f,g) = d_x(f_{k-1},g_{k-1})$.
(d′) If $k > 1$ then $d_x(f,g) > \inf\{d_x(f,f_{k-2}), d_x(g,g_{k-2})\}$.

To prove Theorem 3.1 let $k \in \{1,\dots,h+1\}$ be the smallest integer such that $d_x(f,g) \le d_x(f,f_{k-1})$. Then for $k > 1$ we have $d_x(f,g) > d_x(f,f_{k-2})$, and by Lemma 4.2, $k \le h'+1$ and (a′) holds.

By Lemma 4.4 we get $d_x(f,g) \le d_x(g,g_{k-1})$.

To check (c′), suppose $d_x(f,g) < d_x(f,f_{k-1})$ and $d_x(f,g) < d_x(g,g_{k-1})$. Then by the STI, $d_x(g,f_{k-1}) = d_x(f,g)$ and $d_x(f,g_{k-1}) = d_x(f,g)$. Thus $d_x(f,g) = d_x(g,f_{k-1}) < d_x(g,g_{k-1})$ and applying again the STI to the power series g, f_{k-1} and g_{k-1} we get

$$d_x(f_{k-1},g_{k-1}) = \inf\{d_x(g,f_{k-1}), d_x(g,g_{k-1})\} = d_x(g,f_{k-1}) = d_x(f,g),$$

which proves (c′).

Suppose that $k > 1$. Then $d_x(f,g) > d_x(f,f_{k-2})$ by the definition of k, and $d_x(g,g_{k-2}) = d_x(f,f_{k-2})$ by (a′). This proves (d′). The assertion on the key polynomials follows from Lemma 4.2. ∎

5. Application to polynomial automorphisms. In [vdK] van der Kulk proved a theorem on polynomial automorphisms of the plane generalizing a previous result of Jung [J] to the case of arbitrary characteristic. The proof of van der Kulk is based on a lemma on the intersection multiplicity of branches proved using the Hamburger–Noether expansions (see also [R, Remark 6.3.1]).

As an application of our main result we prove here a property of intersection multiplicities of branches which implies van der Kulk's lemma (see [P] for char $\mathbf{K} = 0$).

PROPOSITION 5.1. *Let* $\{f = 0\}$ *and* $\{g = 0\}$ *be two different branches and let* $\{l = 0\}$ *be a smooth branch. Suppose that* $n = i_0(f,l) < \infty$, $n' = i_0(g,l) < \infty$ *and let* $d = \gcd(n,n')$. *Then* $i_0(f,g) \equiv 0 \pmod{n/d \text{ or } n'/d}$.

Proof. We may assume that $n, n' > 1$ and $l = x$. Let $k > 0$ be the integer as in Corollary 3.3. Then $i_0(f, g) \leq \inf\{e'_{k-1}v_k, e_{k-1}v'_k\}$. We claim that

$$i_0(f, g) \equiv 0 \ (\mathrm{mod}\ e_{k-1} \text{ or } e'_{k-1}). \tag{5.1}$$

In fact, this is clear when $i_0(f, g) = \inf\{e'_{k-1}v_k, e_{k-1}v'_k\}$. If $i_0(f, g) < \inf\{e'_{k-1}v_k, e_{k-1}v'_k\}$ then we infer (5.1) from the second part of Corollary 3.3.

By the first part of Corollary 3.3 and by Remark 3.2 we get

$$\frac{n}{e_i} = \frac{n'}{e'_i} \quad \text{for } i < k. \tag{5.2}$$

From (5.2) we have

$$e_{k-1} \equiv 0 \ (\mathrm{mod}\ n/d) \quad \text{and} \quad e'_{k-1} \equiv 0 \ (\mathrm{mod}\ n'/d). \tag{5.3}$$

Now (5.1) and (5.3) imply the proposition. ∎

Using Proposition 5.1 we will prove the following basic property of polynomial automorphisms of the plane.

THEOREM 5.2 ([vdK, Lemma on p. 36]). *Let $(P, Q) : \mathbf{K}^2 \to \mathbf{K}^2$ be a polynomial automorphism. Then one of the two integers $m = \deg P$, $n = \deg Q$ divides the other.*

Proof. Let C and D be projective curves with affine equations $P = 0$ and $Q = 0$ respectively. Then $\deg D = n$, $\deg C = m$ and each of the curves C, D has exactly one branch at infinity (see [vdK, p. 37]). By Bézout's theorem these branches intersect with multiplicity $i = mn - 1$. The line at infinity cuts the branches of C and D with multiplicities m and n respectively. Thus by Proposition 5.1 we get $i \equiv 0$ (mod m/d or n/d), where $d = \gcd(m, n)$. This implies that m divides n or n divides m, since $i = mn - 1$. ∎

6. Intersection multiplicities of two branches. Let $\overline{\mathbf{v}} = (v_0, \ldots, v_h)$ and $\overline{\mathbf{v}}' = (v'_0, \ldots, v'_{h'})$ be two characteristic sequences. Put $e_i = \gcd(v_0, \ldots, v_i)$, $e'_j = \gcd(v'_0, \ldots, v'_j)$ for $0 \leq i \leq h$ and $0 \leq j \leq h'$. By convention, $v_{h+1} = v'_{h'+1} = \infty$ and $e_{-1} = e'_{-1} = 0$. Let

$$\rho := \max\left\{ i \in \mathbb{N} : \frac{v_j}{v_0} = \frac{v'_j}{v'_0} \text{ for } j \leq i,\ i \leq \min(h, h') \right\}$$

and $I_k := \inf\{e_{k-1}v'_k, e'_{k-1}v_k\}$ for $k = 1, \ldots, \rho + 1$. In particular $I_0 = 0$. Observe that $e_{k-1}v'_k = e'_{k-1}v_k$ for $0 \leq k \leq \rho$ and $I_0 < I_1 < \cdots < I_{\rho+1}$. We put $\mathbb{N}^+ = \{N \in \mathbb{N} : N > 0\}$.

Let f, g run through the irreducible coprime series in $\mathbf{K}[[x, y]]$. Using local quadratic transformations, Bayer [B, Theorem 5] gave an explicit formula for the set $\{i_0(f, g) : \overline{\mathrm{char}}_x f = \overline{\mathbf{v}}, \overline{\mathrm{char}}_x g = \overline{\mathbf{v}}'\}$ in terms of the characteristic sequences $\overline{\mathbf{v}}$ and $\overline{\mathbf{v}}'$.

Hefez [He, proof of Theorem 8.5, pp. 116–117] proved a stronger result. Apart from Bayer's formula, he showed that for a fixed f_0 with $\overline{\mathrm{char}}_x\, f_0 = \overline{\mathbf{v}}$, we have

$$\{i_0(f_0, g) : \overline{\mathrm{char}}_x\, g = \overline{\mathbf{v}}'\} = \{i_0(f, g) : \overline{\mathrm{char}}_x\, f = \overline{\mathbf{v}},\ \overline{\mathrm{char}}_x\, g = \overline{\mathbf{v}}'\}.$$

His proof, based on Puiseux's theorem, works only in characteristic zero. The following theorem is a strong version of Bayer's result, proposed by Hefez. The characteristic of $\mathbf{K}$ is arbitrary.

THEOREM 6.1. *Let $\{f = 0\}$ be a branch such that $\overline{\mathrm{char}}_x\, f = \overline{\mathbf{v}}$. Let $\mathcal{B}$ be the set of branches $\{g = 0\}$ such that $\overline{\mathrm{char}}_x\, g = \overline{\mathbf{v}}'$ and $i_0(f, g) \neq \infty$. Then*

$$\begin{aligned}&\{i_0(f, g) : \{g = 0\} \in \mathcal{B}\}\\ &\qquad = \bigcup_{k=1}^{\rho+1} \{N \in \mathbb{N}^+ : I_{k-1} \le N < I_k \text{ and } N \equiv 0 \pmod{e_{k-1}e'_{k-1}}\}.\end{aligned}$$

COROLLARY 6.2. *Let $\{f = 0\}$ be a branch such that $\overline{\mathrm{char}}_x\, f = \overline{\mathbf{v}}$. Then*

$$\begin{aligned}&\{i_0(f, g) : \overline{\mathrm{char}}_x\, f = \overline{\mathrm{char}}_x\, g \text{ and } i_0(f, g) \neq \infty\}\\ &\qquad = \bigcup_{k=1}^{h+1} \{N \in \mathbb{N}^+ : e_{k-2}v_{k-1} \le N < e_{k-1}v_k \text{ and } N \equiv 0 \pmod{e_{k-1}^2}\}.\end{aligned}$$

COROLLARY 6.3. *Let $\{f = 0\}$ be a branch such that $\overline{\mathrm{char}}_x\, f = \overline{\mathbf{v}}$. Then $N_0 = e_{h-1}v_h$ is the smallest natural number such that for any $N \in \mathbb{N}$ and $N \ge N_0$ there exists an irreducible power series $g \in \mathbf{K}[[x, y]]$ such that $i_0(f, g) = N$.*

To prove Theorem 6.1 we need two lemmas.

LEMMA 6.4. *For any integer k with $1 < k \le \rho + 1$, there exists an irreducible power series $g \in \mathbf{K}[[x, y]]$ such that $i_0(f, g) = I_{k-1}$.*

Proof. Suppose that $k > 1$ and let $f_0, \dots, f_{k-2}$ be key polynomials of f. Let $g_i = f_i$ for $i = 0, \dots, k-2$ and $g_i = g_{i-1}^{n'_i} + \xi_i x^{a_{i,0}} g_0^{a_{i,1}} \cdots g_{i-2}^{a_{i,i-1}}$ for $i = k-1, \dots, h'$, where $\xi_i \in \mathbf{K} \setminus \{0\}$, $n'_i = e'_{i-1}/e'_i$ and $a_{i,0}v'_0 + \cdots + a_{i,i-1}v'_{i-1} = n'_i v'_i$. By (key 2) we see that $g_0, \dots, g_{h'}$ are key polynomials of $g := g_{h'}$ and $\overline{\mathrm{char}}_x\, f_i = (v_0/e_i, \dots, v_i/e_i) = (v'_0/e'_i, \dots, v'_i/e'_i)$ for $i = 0, \dots, k-2$.

We have

$$\begin{aligned}&i_0(f, x^{a_{k-1,0}} g_0^{a_{k-1,1}} \cdots g_{k-3}^{a_{k-1,k-2}})\\ &\qquad = a_{k-1,0}i_0(f, x) + a_{k-1,1}i_0(f, g_0) + \cdots + a_{k-1,k-2}i_0(f, g_{k-3})\\ &\qquad = a_{k-1,0}i_0(f, x) + a_{k-1,1}i_0(f, f_0) + \cdots + a_{k-1,k-2}i_0(f, f_{k-3})\\ &\qquad = a_{k-1,0}v_0 + a_{k-1,1}v_1 + \cdots + a_{k-1,k-2}v_{k-2}\end{aligned}$$

$$
\begin{aligned}
&= e_{k-2}\left(a_{k-1,0}\frac{v_0}{e_{k-2}} + a_{k-1,1}\frac{v_1}{e_{k-2}} + \cdots + a_{k-1,k-2}\frac{v_{k-2}}{e_{k-2}}\right)\\
&= e_{k-2}\left(a_{k-1,0}\frac{v'_0}{e'_{k-2}} + a_{k-1,1}\frac{v'_1}{e'_{k-2}} + \cdots + a_{k-1,k-2}\frac{v'_{k-2}}{e'_{k-2}}\right)\\
&= \frac{e_{k-2}}{e'_{k-2}} n'_{k-1}v'_{k-1} = \frac{n'_{k-1}}{e'_{k-2}} e_{k-2}v'_{k-1} = \frac{n'_{k-1}}{e'_{k-2}} e'_{k-2}v_{k-1}\\
&= n'_{k-1}v_{k-1} = n'_{k-1}i_0(f, f_{k-2}) = i_0(f, f_{k-2}^{n'_{k-1}}) = i_0(f, g_{k-2}^{n'_{k-1}}).
\end{aligned}
$$

Suppose that ξ_{k-2} is generic. Then $i_0(f, g_{k-1}) = i_0(f, g_{k-2}^{n'_{k-1}}) = n'_{k-1}v_{k-1}$. We get

$$d_x(f, g_{k-1}) = \frac{n'_{k-1}v_{k-1}}{v_0(v'_0/e'_{k-1})} = \frac{e'_{k-1}n'_{k-1}v_{k-1}}{v_0v'_0} = \frac{e'_{k-2}v_{k-1}}{v_0v'_0} = \frac{I_{k-1}}{v_0v'_0}$$

and

$$d_x(g, g_{k-1}) = \frac{v'_k}{v'_0(v'_0/e'_{k-1})} = \frac{e'_{k-1}v'_k}{v'_0v'_0} = \frac{e_{k-1}v'_k}{v_0v'_0} = \frac{I_k}{v_0v'_0}.$$

Therefore $d_x(f, g_{k-1}) < d_x(g, g_{k-1})$, thus $d_x(f, g) = d_x(f, g_{k-1})$, that is,

$$\frac{i_0(f, g)}{v_0v'_0} = \frac{I_{k-1}}{v_0v'_0} \quad \text{and} \quad i_0(f, g) = I_{k-1}. \ \blacksquare$$

LEMMA 6.5. *Let $\{f = 0\}$ be a branch with $\overline{\mathrm{char}}_x f = \overline{\mathbf{v}}$. Let N be a positive integer number such that $N > e_{h-1}v_h$. Then there exists a branch $\{g = 0\}$ such that $i_0(f, g) = N$ and $\overline{\mathrm{char}}_x g = \overline{\mathrm{char}}_x f$. Moreover $f_0, \ldots, f_{h-1}$ are key polynomials of g.*

Proof. Let c be the conductor of $\Gamma(f)$. Then $c = (n_1 - 1)v_1 + \cdots + (n_h - 1)v_h - v_0 + 1 = (n_1v_1 - v_1) + \cdots + (n_hv_h - v_h) - v_0 + 1 < (v_2 - v_1) + (v_3 - v_2) + \cdots + (e_{h-1}v_h - v_h) - v_0 + 1 = e_{h-1}v_h - v_0 - v_1 + 1 < e_{h-1}v_h$ and we may write $N = a_0v_0 + \cdots + a_hv_h$, where $a_i \in \mathbb{N}$ with $a_0 > 0$. Let f_k be a key polynomial of f for $k = 0, \ldots, h-1$. Put

$$g := f + x^{a_0}f_0^{a_1}\cdots f_{h-1}^{a_h}.$$

Then $i_0(x, g) = i_0(x, f)$ since $a_0 > 0$, and $i_0(f, g) = i_0(f, x^{a_0}f_0^{a_1}\cdots f_{h-1}^{a_h}) = a_0v_0 + \cdots + a_hv_h = N$. By the Abhyankar–Moh irreducibility criterion (see Section 2), g is irreducible and $\overline{\mathrm{char}}_x g = \overline{\mathrm{char}}_x f$.

Observe that $d_x(f, g) = N/v_0^2 > e_{h-1}v_h/v_0^2 \geq e_kv_{k+1}/v_0^2 = d_x(f, f_k)$. Thus, by the STI, we have $d_x(f_k, g) = d_x(f_k, f)$, which implies $i_0(f_k, g) = i_0(f_k, f) = v_{k+1}$. Therefore f_k is a key polynomial of g for $k = 0, \ldots, h-1$. $\blacksquare$

Proof of Theorem 6.1. The inclusion "$\subset$" follows from Corollary 3.3. Let $N > 0$ be an integer such that $I_{k-1} \leq N < I_k$ and $N \equiv 0 \pmod{e_{k-1}e'_{k-1}}$, where $1 \leq k \leq \rho + 1$. We have to prove that there exists an irreducible power series $g \in \mathbf{K}[[x, y]]$ such that $\overline{\mathrm{char}}_x g = \overline{\mathbf{v}}'$ and $i_0(f, g) = N$. If $N = I_{k-1}$ then the theorem follows from Lemma 6.4. Suppose that $I_{k-1} < N < I_k$. Since $N \equiv 0 \pmod{e_{k-1}e'_{k-1}}$, we may write $N = e_{k-1}e'_{k-1}N_{k-1}$ for some $N_{k-1} \in \mathbb{N}$. Let $f_0, \dots, f_{k-1}$ be key polynomials of f, where $\overline{\mathrm{char}}_x\, f_i = (v_0/e_i, \dots, v_i/e_i) = (v'_0/e'_i, \dots, v'_i/e'_i)$ for $i = 0, \dots, k-1$ by the definition of ρ. Put $g_i := f_i$ for $i = 0, \dots, k-2$.

Claim 1. *There exists an irreducible power series $g_{k-1} \in \mathbf{K}[[x,y]]$ such that $\overline{\mathrm{char}}_x\, g_{k-1} = \overline{\mathrm{char}}_x\, f_{k-1}$ and $i_0(f_{k-1}, g_{k-1}) = N_{k-1}$.*

The claim follows from

$$\begin{aligned}\gcd\left(\frac{v_0}{e_{k-1}}, \dots, \frac{v_{k-2}}{e_{k-1}}\right)\frac{v_{k-1}}{e_{k-1}} &= \frac{e_{k-2}v_{k-1}}{e_{k-1}^2} = \frac{e'_{k-2}v_{k-1}}{e'_{k-1}e_{k-1}} = \frac{I_{k-1}}{e_{k-1}e'_{k-1}} \\ &< \frac{N}{e_{k-1}e'_{k-1}} = N_{k-1},\end{aligned}$$

and Lemma 6.5 applied to $\{f_{k-1} = 0\}$.

Claim 2. $i_0(f, g_{k-1}) = e_{k-1}N_{k-1}$.

Indeed, if $v_{\rho+1} = \infty$ then f_ρ is the distinguished polynomial associated with f. Therefore $i_0(f, g_\rho) = i_0(f_\rho, g_\rho) = N_\rho$ $(= e_\rho N_\rho$ since $e_\rho = 1)$ by Claim 1.

Assume that $v_{\rho+1} \neq \infty$. Then

$$\begin{aligned}d_x(f_{k-1}, g_{k-1}) &= \frac{N_{k-1}}{(v_0/e_{k-1})(v'_0/e'_{k-1})} = \frac{N}{v_0v'_0} \\ &< \frac{I_k}{v_0v'_0} \leq \frac{e'_{k-1}v_k}{v_0v'_0} = \frac{e_{k-1}v_k}{v_0^2} = d_x(f, f_{k-1}).\end{aligned}$$

Therefore by the STI we have

$$d_x(f, g_{k-1}) = d_x(f_{k-1}, g_{k-1}) = \frac{N}{v_0v'_0} \quad \text{and} \quad \frac{i_0(f, g_{k-1})}{v_0(v'_0/e'_{k-1})} = \frac{e_{k-1}e'_{k-1}N_{k-1}}{v_0v'_0},$$

which implies $i_0(f, g_{k-1}) = e_{k-1}N_{k-1}$.

Let us now finish the proof of Theorem 6.1. Let $g = g_{h'}$. We will check that $d_x(f, g) = d_x(f, g_{k-1})$. Firstly suppose that $v'_{\rho+1} = \infty$. Then g_ρ is the distinguished polynomial associated with g and $d_x(f, g) = d_x(f, g_\rho)$. If $v'_{\rho+1} \neq \infty$ then

$$d_x(f, g_{k-1}) = \frac{e_{k-1}N_{k-1}}{v_0(v'_0/e'_{k-1})} = \frac{N}{v_0v'_0},$$

$$d_x(g, g_{k-1}) = \frac{v'_k}{v'_0(v'_0/e'_{k-1})} = \frac{e'_{k-1}v'_k}{v'_0v'_0} = \frac{e_{k-1}v'_k}{v_0v'_0} \geq \frac{I_k}{v_0v'_0} > \frac{N}{v_0v'_0} = d_x(f, g_{k-1}),$$

and by the STI, $d_x(f, g) = d_x(f, g_{k-1})$. Therefore $\frac{i_0(f,g)}{v_0v'_0} = \frac{N}{v_0v'_0}$ and we get $i_0(f, g) = N$. ∎

7. A property of the logarithmic distance. Recall that the logarithmic distance $d(f,g)$ among two branches $\{f=0\}$ and $\{g=0\}$ is given by

$$d(f,g)=\frac{i_0(f,g)}{\operatorname{ord} f \operatorname{ord} g}.$$

Observe that $d(f,g)=d_x(f,g)$ when $\{x=0\}$ is transverse to $\{f=0\}$ and $\{g=0\}$.

If $\{f=0\}$ and $\{g=0\}$ have no common tangent then $d(f,g)=1$. The next theorem generalizes [Ab-Al-G, Theorem 2.7] to arbitrary characteristic.

THEOREM 7.1. *Let $f\in \mathbf{K}[[x,y]]$ be an irreducible power series and let $R>1$ be a rational number. Then there exists an irreducible power series $g\in \mathbf{K}[[x,y]]$ such that $d(f,g)=R$.*

Proof. Let $\overline{\operatorname{char}}_x f=(v_0,\dots,v_h)$, where $v_0<v_1$. Fix a rational number $R>1$. We distinguish two cases:

CASE 1: There exists an integer k, $1\le k\le h$, such that $R=e_{k-1}v_k/v_0^2$. Then for the $(k-1)$th key polynomial f_{k-1} of f we have $i_0(f,f_{k-1})=v_k$, $\operatorname{ord} f=v_0$, $\operatorname{ord} f_{k-1}=v_0/e_{k-1}$ and $d(f,f_{k-1})=R$.

CASE 2: The number R is different from $e_{l-1}v_l/v_0^2$ for $l=1,\dots,h$. Then there exists a unique k, $1\le k\le h+1$, such that $e_{k-2}v_{k-1}/v_0^2<R<e_{k-1}v_k/v_0^2$ (recall that $e_{-1}=0$). Write

$$R=\frac{r}{(v_0/e_{k-1})^2 s},$$

where $\gcd(r,s)=1$. Let $s>1$. Put

$$(v_0',\dots,v_k')=\left(s\frac{v_0}{e_{k-1}},\dots,s\frac{v_{k-1}}{e_{k-1}},r\right).$$

We check that

(1) $(v_0',\dots,v_k')$ is a characteristic sequence,
(2) $\frac{v_1'}{e_0'}=\frac{v_1}{e_0},\dots,\frac{v_{k-1}'}{e_0'}=\frac{v_{k-1}}{e_0}$ and $\frac{v_k'}{e_0'}<\frac{v_k}{e_0}$, where $e_i':=\gcd(v_0',\dots,v_i')$.

By Theorem 6.1 there exists an irreducible g such that $\overline{\operatorname{char}}_x g=(v_0',\dots,v_k')$ and $i_0(f,g)=\inf\{e_{k-1}'v_k,e_{k-1}v_k'\}$. Therefore

$$d(f,g)=\frac{i_0(f,g)}{v_0v_0'}=\inf\left\{\frac{e_{k-1}'v_k}{e_0'e_0},\frac{e_{k-1}v_k'}{e_0e_0'}\right\}=\frac{e_{k-1}v_k'}{e_0e_0'}=R.$$

Now let $s=1$. Then $R=re_{k-1}^2/v_0^2$ and $e_{k-2}v_{k-1}<re_{k-1}^2<e_{k-1}v_k$. By Corollary 6.2 there exists an irreducible power series g such that $\operatorname{ord} g=\operatorname{ord} f=v_0$ and $i_0(f,g)=re_{k-1}^2$. Clearly $d(f,g)=R$. ∎

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