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CONTACT EXPONENT AND THE MILNOR NUMBER OF PLANE CURVE SINGULARITIES

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Abstract. We investigate properties of the contact exponent (in the sense of Hironaka [Hi]) of plane algebroid curve singularities over algebraically closed fields of arbitrary characteristic. We prove that the contact exponent is an equisingularity invariant and give a new proof of the stability of the maximal contact. Then we prove a bound for the Milnor number and determine the equisingularity class of algebroid curves for which this bound is attained. We do not use the method of Newton's diagrams. Our tool is the logarithmic distance developed in [GB-P1].

Introduction

Let C be a plane algebroid curve of multiplicity $m(C)$ defined over an algebraically closed field K. To calculate the number of infinitely near $m(C)$-fold points, Hironaka [Hi] (see also [B-K] or [T2]) introduced the concept of contact exponent $d(C)$ and study its properties using Newton's diagrams.

In this note we prove an explicit formula for a generalization of contact exponent (Section 2, Theorem 2.3) using the logarithmic distance on the set of branches. Then we give a new proof of the stability of maximal contact (Section 3, Theorem 3.7) without resorting to Newton's diagrams. In Section 4 we define the Milnor number $\mu(C)$ in the case of arbitrary characteristic (see [M-W] and [GB-P2]), prove the bound $\mu(C) \geq (d(C)m(C)-1)(m(C)-1)$ and characterize the singularities for which the bound is attained. In Section 5 we reprove the formulae for the contact exponents of higher order (see [LJ] and [C]). Section 6 is devoted to the relation between polar invariants and the contact exponent in characteristic zero.

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1. Preliminaries

Let $K[[x,y]]$ be the ring of formal power series with coefficients in an algebraically closed field K of arbitrary characteristic. For any non-zero power series $f = f(x,y) = \sum_{i,j} c_{ij}x^i y^j \in K[[x,y]]$ we define its *order* as ord $f = \inf\{i+j \ : \ c_{ij} \neq 0\}$ and its *initial form* as in$f = \sum_{i+j=n} c_{ij}x^i y^j$, where $n =$ ord f. We let $(f,g)_0 = \dim_K K[[x,y]]/(f,g)$, and call $(f,g)_0$ the *intersection number* of f and g, where (f,g) denotes the ideal of $K[[x,y]]$ generated by f and g.

Let f be a nonzero power series without constant term. An *algebroid curve* $C : \{f = 0\}$ is defined to be the ideal generated by f in $K[[x,y]]$. The *multiplicity* of C is $m(C) = \text{ord } f$. Let $\mathbb{P}^1(K)$ denotes the projective line over K. The *tangent cone* of C is by definition cone$(C) = \{(a:b) \in \mathbb{P}^1(K) \ : \ \text{in}f(a,b) = 0\}$.

The curve $C : \{f = 0\}$ is *reduced* (resp. *irreducible*) if the power series f has no multiple factors (resp. is irreducible). Irreducible curves are called *branches*. If $\sharp$cone$(C) = 1$ then the curve $C : \{f = 0\}$ is called *unitangent*. Any irreducible curve is unitangent. For $C : \{f = 0\}$ and $D : \{g = 0\}$ we put $(C,D)_0 = (f,g)_0$. Then $(C,D)_0 \geq m(C)m(D)$, with equality if and only if the tangent cones of C and D are disjoint.

For any sequence $C_i : \{f_i = 0 \ : \ 1 \leq i \leq k\}$ of curves we put $C = \bigcup_{i=1}^k C_i : \{f_1 \cdots f_k = 0\}$. If C_i are irreducible and $C_i \neq C_j$ for $i \neq j$ then we call C_i the *irreducible components* of C.

Consider an irreducible power series $f \in K[[x,y]]$. The set

$$\Gamma(C) = \Gamma(f) := \{(f,g)_0 \ : \ g \in K[[x,y]], \ \ g \not\equiv 0 \ (\text{mod } f)\}$$

is the *semigroup* associated with $C : \{f = 0\}$. Note that $\min(\Gamma(C)\backslash\{0\}) = m(C)$. It is well-known that $\gcd(\Gamma(C)) = 1$.

The branch C is *smooth* (that is its multiplicity equals 1) if and only if $\Gamma(C) = \mathbb{N}$.

Two branches $C : \{f = 0\}$ and $D : \{g = 0\}$ are *equisingular* if $\Gamma(C) = \Gamma(D)$.

Two reduced curves $C : \{f = 0\}$ and $D : \{g = 0\}$ are equisingular if and only if f and g have the same number r of irreducible factors and there are factorizations $f = f_1 \cdots f_r$ and $g = g_1 \cdots g_r$ such that

(1) the branches $C_i : \{f_i = 0\}$ and $D_i : \{g_i = 0\}$ are equisingular for $i \in \{1, \dots, r\}$,
and
(2) $(C_i, C_j)_0 = (D_i, D_j)_0$ for any $i, j \in \{1, \dots, r\}$.

A function $C \mapsto I(C)$ defined on the set of all reduced curves is an *equisingularity invariant* if $I(C) = I(D)$ for equisingular curves C and D. Note that the multiplicity $m(C)$, the number of branches $r(C)$ and the number of tangents $t(C)$ (which is the cardinality of the cone(C)) of the reduced curve C are equisingularity invariants.

For any reduced curve $C : \{f = 0\}$ we put $\mathcal{O}_C = K[[x, y]]/(f)$ and $\overline{\mathcal{O}}_C$ its integral closure. Let $\mathcal{C} = \overline{\mathcal{O}}_C : \mathcal{O}_C$ be the *conductor* of $\overline{\mathcal{O}}_C$ in $\mathcal{O}_C$. The number $c(C) = \dim_K \overline{\mathcal{O}}_C/\mathcal{C}$ is the *degree of the conductor*. If C is a branch then $c(C)$ equals to the smallest element of $\Gamma(C)$ such that $c(C) + N \in \Gamma(C)$ for all $N \in \mathbb{N}$ (see [C, Chapter IV]).

Suppose that C is a branch. Let $(v_0, v_1, \dots, v_g)$ be the *minimal system of generators* of $\Gamma(C)$ defined by the following conditions:

(3) $v_0 = \min(\Gamma(C)\backslash\{0\}) = m(C)$.
(4) $v_k = \min(\Gamma(C)\backslash\mathbb{N}v_0 + \dots + \mathbb{N}v_{k-1})$, for $k \in \{1, \dots, g\}$.
(5) $\Gamma(C) = \mathbb{N}v_0 + \dots + \mathbb{N}v_g$.

In what follows we write $\Gamma(C) = \langle v_0, v_1, \dots, v_g\rangle$ when $v_0 < v_1 < \dots < v_g$ is the increasing sequence of minimal system of generators of $\Gamma(C)$.

Since $\gcd(\Gamma(C)) = 1$ the sequence $v_0, \dots, v_g$ is well-defined. Let $e_k := \gcd(v_0, \dots, v_k)$ for $0 \le k \le g$. We define the *Zariski pairs* $(m_k, n_k) = \left(\frac{v_k}{e_k}, \frac{e_{k-1}}{e_k}\right)$ for $1 \le k \le g$. One has $c(C) = \sum_{k=1}^{g} (n_k - 1)\, v_k - v_0 + 1$ (see[GB-P1, Corollary 3.5]).

If K is a field of charateristic zero the Zariski pairs determine the *Puiseux pairs* and vice versa (see [T2, pp. 19, 47]).

If $\Gamma(C) = \langle v_0, v_1, \dots, v_g\rangle$ then the sequence $(v_i)_i$ is *strongly increasing*, that is $n_{i-1}v_{i-1} < v_i$ for $i \in \{2, \dots, g\}$.

Let $C : \{f = 0\}$ be a reduced unitangent curve of multiplicity n. Let us consider two possible cases:

(i) $f = c(y - ax)^n +$ higher order terms, where $a, c \in K$, $c \neq 0$ and
(ii) $f = cx^n +$ higher order terms, $c \in K\backslash\{0\}$.

We associate with C a power series $f_1 = f_1(x_1, y_1) \in K[[x_1, y_1]]$ by putting $f_1(x_1, y_1) = x_1^{-n} f(x_1, ax_1 + x_1y_1)$ in the case (i) and $f_1(x_1, y_1) = y_1^{-n} f(x_1y_1, y_1)$ otherwise. The *strict quadratic transform* of $C : \{f = 0\}$ is the curve $\widehat{C} : \{f_1 = 0\}$.

Obviously $m(\widehat{C}) \le m(C)$. If $C = \bigcup_{i=1}^k C_i$ is a unitangent curve then C_i are unitangent and $\widehat{C} = \bigcup_{i=1}^k \widehat{C}_i$.

The following lemma is a particular case of a theorem due to Angermüller [Ang, Lemma II.2.1].

Lemma 1.1. *Let C be a singular branch. Then the strict quadratic transform $\widehat{C}$ of C is also a plane branch. If $\Gamma(C) = \langle v_0, \dots, v_g\rangle$ then*

- $\Gamma(\widehat{C}) = \langle v_0, v_1 - v_0, \dots\rangle$ *if* $v_0 < v_1 - v_0$
  or
- $\min(\Gamma(\widehat{C})\backslash\{0\}) = v_1 - v_0$ *if* $v_1 - v_0 < v_0$.

2. Logarithmic distance

A *log-distance* δ associates with any two branches C, D a number $\delta(C,D) \in \mathbb{R}_+ \cup \{+\infty\}$ such that for any branches C, D and E we have:

(δ_1) $\delta(C,D) = \infty$ if and only if $C = D$,
(δ_2) $\delta(C,D) = \delta(D,C)$,
(δ_3) $\delta(C,D) \geq \inf\{\delta(C,E), \delta(E,D)\}$.

Note that if $\delta(C,E) \neq \delta(E,D)$ then $\delta(C,D) = \inf\{\delta(C,E), \delta(E,D)\}$.

If C and D are reduced curves with irreducible components C_i and D_j then we set $\delta(C,D) := \inf_{\,i,j}\{\delta(C_i, D_j)\}$.

If δ is a log-distance then $\Delta := \frac{1}{\delta}$ (by convention $\frac{1}{+\infty} = 0$) is an *ultrametric* (see [GB-GP-PP, Definition 41]) on the set of branches and vice versa: if Δ is an ultrametric then $\frac{1}{\Delta}$ is a log-distance.

Examples 2.1.

(1) *The order of contact of branches* $d(C,D) = \frac{(C,D)_0}{m(C)m(D)}$ *is a log-distance (see* [GB-P1, Corollary 2.9]*).*
(2) *The minimum number of quadratic transformations* $\gamma(C,D)$ *necessary to separate* C *from* D *is a log-distance (see* [W, Theorem 3]*).*

Let δ be a log-distance.

Lemma 2.2. *If* C *has* $r > 1$ *branches* C_i *and* D *is any branch then* $\delta(C,D) \leq \inf_{i,j}\{\delta(C_i, C_j)\}$.

Proof. Let i_0, j_0 be such that $\inf_{i,j}\{\delta(C_i,C_j)\} = \delta(C_{i_0}, C_{j_0})$. Then $\delta(C,D) = \inf_{1\leq i\leq r}\{\delta(C_i,D)\} \leq \inf\{\delta(C_{i_0},D), \delta(C_{j_0},D)\}$ and using (δ3) we get $\delta(C,D) \leq \delta(C_{i_0}, C_{j_0})$, which proves the lemma. □

Let C be a reduced curve. For every non-empty family of branches $\mathcal{B}$ we put

$$\delta(C,\mathcal{B}) := \sup\{\delta(C,W) \ : \ W \in \mathcal{B}\}.$$

Note that $\delta(C,\mathcal{B}) = +\infty$ if $C \in \mathcal{B}$. In what follows we assume the following condition

(*) for any branch C there exists $W_0 \in \mathcal{B}$ such that $\delta(C,\mathcal{B}) = \delta(C, W_0)$,
we say that W_0 has *maximal* δ*-contact* with C.

We will prove the following

Theorem 2.3. *Let* C *be a reduced curve with* $r > 1$ *branches* C_i *and let* $\mathcal{B}$ *be a family of branches such that the condition (*) holds.*
Then

$$\delta(C,\mathcal{B}) = \inf\{\inf_i\{\delta(C_i,\mathcal{B})\}, \inf_{i,j}\{\delta(C_i,C_j)\}\}.$$

Moreover, there exists $i_0 \in \{1,\dots,r\}$ such that if a branch $W \in \mathcal{B}$ has maximal δ-contact with C_{i_0} then it has maximal δ-contact with C.

Proof. Set $\delta^*(C,\mathcal{B}) = \inf\{\inf_i \delta(C_i,\mathcal{B}), \inf_{i,j}\delta(C_i,C_j)\}$.

The inequality $\delta(C,\mathcal{B}) \le \delta^*(C,\mathcal{B})$ follows from Lemma 2.2 and from the definition of $\delta(C_i,\mathcal{B})$. Thus to prove the result let us consider two cases:

First case: $\inf_i\{\delta(C_i,\mathcal{B})\} \le \inf_{i,j}\{\delta(C_i,C_j)\}$.

Let $i_0 \in \{1,\dots,r\}$ be such that $\delta(C_{i_0},\mathcal{B}) = \inf_i\{\delta(C_i,\mathcal{B})\}$. Then, we have

$$\delta(C_{i_0},\mathcal{B}) = \delta^*(C,\mathcal{B}). \tag{1}$$

Let $W \in \mathcal{B}$ such that $\delta(C_{i_0},W) = \delta(C_{i_0},\mathcal{B})$. We claim that

$$\delta(C_{i_0},W) \le \delta(C_i,W) \ \text{ for all } i \in \{1,\dots,r\}. \tag{2}$$

To obtain a contradiction suppose that (2) does not hold. Thus there is $i_1 \in \{1,\dots,r\}$ such that

$$\delta(C_{i_1},W) < \delta(C_{i_0},W). \tag{3}$$

Applying Property (δ_3) to the branches C_{i_0}, C_{i_1} and W we get

$$\delta(C_{i_1},W) = \delta(C_{i_0},C_{i_1}). \tag{4}$$

On the other hand, in the case under consideration we have

$$\delta(C_{i_0},\mathcal{B}) = \inf_i\{\delta(C_i,\mathcal{B})\} \le \delta(C_{i_0},C_{i_1}). \tag{5}$$

Therefore by (5), (4) and (3) we get $\delta(C_{i_0},\mathcal{B}) \le \delta(C_{i_1},W) < \delta(C_{i_0},W)$, which contradicts the definition of $\delta(C_{i_0},\mathcal{B})$.

Now, using (2) and (1), we compute

$$\delta(C,W) = \inf\{\delta(C_{i_0},W), \inf_{i\ne i_0}(\delta(C_i,W))\} = \delta(C_{i_0},W) = \delta(C_{i_0},\mathcal{B}) = \delta^*(C,\mathcal{B}),$$

which proves the theorem in the first case.

Second case: $\inf_i\{\delta(C_i,\mathcal{B})\} > \inf_{i,j}\{\delta(C_i,C_j)\}$.

Let i_0, j_0 be such that $\delta(C_{i_0},C_{j_0}) = \inf_{i,j}\delta(C_i,C_j) = \delta^*(C,\mathcal{B})$.

Let $W \in \mathcal{B}$ such that $\delta(C_{i_0},W) = \delta(C_{i_0},\mathcal{B})$. We claim that

$$\delta(C_{i_0},C_{j_0}) \le \delta(C_i,W) \ \text{ for all } i \in \{1,\dots,r\} \ \text{ with equality for } i = j_0. \tag{6}$$

First observe that in the case under consideration we have

$$\delta(C_{i_0},C_{j_0}) < \delta(C_{i_0},\mathcal{B}) = \delta(C_{i_0},W). \tag{7}$$

Fix $i \in \{1,\dots,r\}$. If $\delta(C_{i_0},W) \le \delta(C_i,W)$ then (6) follows from (7). If $\delta(C_i,W) < \delta(C_{i_0},W)$ then by Property (δ_3) applied to the branches C_i, C_{i_0} and W we get $\delta(C_i,W) = \delta(C_i,C_{i_0}) \ge \inf_{i,j}\{\delta(C_i,C_j)\} = \delta(C_{i_0},C_{j_0})$. In particular for $i = j_0$, $\delta(C_{j_0},W) = \delta(C_{j_0},C_{i_0}) = \delta(C_{i_0},C_{j_0})$.

Now by the definition of $\delta(C,W)$ and inequalities (6) and (7) we get:

$$\begin{aligned}\delta(C,W) &= \inf\{\delta(C_{i_0},W),\delta(C_{j_0},W),\inf_{i\neq i_0,j_0}\delta(C_i,W)\}\\ &= \delta(C_{j_0},W)=\delta(C_{i_0},C_{j_0})=\delta^*(C,\mathcal{B}),\end{aligned}$$

which proves the theorem in the second case. □

Proposition 2.4. *Let C and D be two branches. Then*

(1) *If there exists a branch of $\mathcal{B}$ which has maximal δ-contact with C and D then $\delta(C,D)\geq\inf\{\delta(C,\mathcal{B}),\delta(D,\mathcal{B})\}$ with equality if $\delta(C,\mathcal{B})\neq\delta(D,\mathcal{B})$.*
(2) *If there does not exist such a branch and U has maximal δ-contact with C and V has maximal δ-contact with D then $\delta(C,D)=\delta(U,V)<\inf\{\delta(C,\mathcal{B}),\delta(D,\mathcal{B})\}$.*

Proof. (see [GB-L-P, Proposition 2.2] for $\delta=d$). If there exists a branch $W\in\mathcal{B}$ such that $\delta(W,C)=\delta(C,\mathcal{B})$ and $\delta(W,D)=\delta(D,\mathcal{B})$ then we get the first part of the proposition by using Property ($\delta3$) to the branches C, D and W. In order to check the second part suppose that such a branch does not exist. Let $U,V\in\mathcal{B}$ such that $\delta(U,C)=\delta(C,\mathcal{B})$ and $\delta(V,D)=\delta(D,\mathcal{B})$. By hypothesis $\delta(C,V)<\delta(C,\mathcal{B})=\delta(C,U)$ and $\delta(D,U)<\delta(D,\mathcal{B})=\delta(D,V)$. According to ($\delta3$) we get $\delta(U,V)=\inf\{\delta(C,V),\delta(C,U)\}=\delta(C,V)$ and $\delta(U,V)=\inf\{\delta(D,U),\delta(D,V)\}=\delta(D,U)$ thus

$$\delta(C,V)=\delta(D,U)=\delta(U,V). \tag{8}$$

Without loss of generality we can suppose that $\delta(C,\mathcal{B})\leq\delta(D,\mathcal{B})$. Since $\delta(C,V)<\delta(C,\mathcal{B})$ so $\delta(C,V)<\delta(D,\mathcal{B})=\delta(D,V)$ and using ($\delta3$) we get

$$\delta(C,D)=\inf\{\delta(C,V),\delta(D,V)\}=\delta(C,V). \tag{9}$$

From (8) and (9) it follows that $\delta(C,D)=\delta(U,V)$. Moreover $\delta(C,D)<\inf\{\delta(C,\mathcal{B}),\delta(D,\mathcal{B})\}$ and we are done. □

Proposition 2.5. *Let C be a reduced curve with $r>1$ branches C_i and let D be a branch. Suppose that $\delta(C,D)<\inf\{\sigma,\inf_{i,j}\{\delta(C_i,C_j)\}\}$, where σ is a real number. Then $\delta(C_i,D)<\sigma$, for $i\in\{1,\dots,r\}$.*

Proof. By definition we have $\delta(C,D)=\inf_{i=1}^r\{\delta(C_i,D)\}$. Thus there exists $i_0\in\{1,\dots,r\}$ such that $\delta(C,D)=\delta(C_{i_0},D)$. Fix $j_0\in\{1,\dots,r\}$. By hypothesis $\delta(C_{i_0},D)<\delta(C_{i_0},C_{j_0})$ and after (δ_3) we have $\delta(C_{i_0},D)=\delta(C_{j_0},D)<\delta(C_{i_0},C_{j_0})$. Now $\delta(C_{j_0},D)=\delta(C_{i_0},D)=\delta(C,D)<\sigma$ and we are done since $j_0\in\{1,\dots,r\}$ is arbitrary. □

Corollary 2.6. *Let C be a reduced curve with $r>1$ branches C_i and let $\mathcal{B}$ be a family of branches such that the condition (*) holds. If $\delta(C,W)<\delta(C,\mathcal{B})$ for a branch $W\in\mathcal{B}$ then $\delta(C_i,W)<\delta(C_i,\mathcal{B})$, for $i\in\{1,\dots,r\}$.*

3. The contact exponent

Recall that $d(C,D)=\frac{(C,D)_0}{m(C)m(D)}$ is a log-distance (see Example 2.1 (1)).

If C and D are reduced curves with irreducible components C_i and D_j then we set $d(C,D)=\inf_{i,j}\{d(C_i,D_j)\}$.

Lemma 3.1. *If C has $r>1$ branches C_i and D is any branch then*

(1) $d(C,D)\leq \inf_{i,j}\{d(C_i,C_j)\}$,
(2) $d(C,D)\leq \frac{(C,D)_0}{m(C)m(D)}$ *with equality if* $d(C,D)<\inf_{i,j}\{d(C_i,C_j)\}$.

Proof. The first part of the lemma follows from Lemma 2.2, for $\delta=d$. In order to check the second part let us observe that

$$\begin{aligned}(C,D)_0 &= \sum_{i=1}^{r}(C_i,D)_0=\sum_{i=1}^{r}d(C_i,D)m(C_i)m(D)\geq\sum_{i=1}^{r}d(C,D)m(C_i)m(D)\\ &= d(C,D)m(C)m(D),\end{aligned}$$

so $d(C,D)\leq\frac{(C,D)_0}{m(C)m(D)}$ with equality if and only if $d(C,D)=d(C_i,D)$ for all $i\in\{1,\ldots,r\}$.

Suppose that $d(C,D)<\inf{}_{i,j}\{d(C_i,C_j)\}$. By definition there is $i_0\in\{1,\ldots,r\}$ such that $d(C,D)=d(C_{i_0},D)$, so $d(C_{i_0},D)<d(C_{i_0},C_j)$ for all $j\in\{1,\ldots,r\}$. Applying ($\delta 3$) ($\delta=d$) to C_{i_0}, D and C_j we get

$$d(C_j,D)=\inf\{d(C_{i_0},D),d(C_j,C_{i_0})\}=d(C_{i_0},D)=d(C,D)\quad\text{for all } j,$$

so $d(C,D)=\frac{(C,D)_0}{m(C)m(D)}$. □

Now we put for any reduced curve C:

$$d(C):=\sup\{d(C,W)\ :\ W\ \text{ runs over all smooth branches}\}$$

and call $d(C)$ the *contact exponent* of C (see [Hi, Definition 1.5] where the term *characteristic exponent* is used). We say that a smooth branch W has *maximal contact* with C if $d(C,W)=d(C)$.

Observe that $d(C)=+\infty$ if C is a smooth branch.

Lemma 3.2. *Let C be a singular branch with $\Gamma(C)=\langle v_0,v_1,\ldots,v_g\rangle$. Then there exists a smooth branch W_0 such that $(C,W_0)_0=v_1$. Moreover, $d(C)=\frac{v_1}{v_0}$ and W_0 has maximal contact with C.*

Proof. See [GB-P1, Proposition 3.6] or [Ang, Folgerung II.1.1] for the first part of the lemma. To check the second part, let W be a smooth branch. We have $d(C,W_0)=\frac{v_1}{v_0}\notin\mathbb{N}$ and $d(W,W_0)=(W,W_0)_0\in\mathbb{N}$. Therefore $d(C,W_0)\neq d(W,W_0)$ and $d(C,W)=\inf\{d(C,W_0),d(W,W_0)\}\leq d(C,W_0)$. □

Proposition 3.3. *Let C be a reduced curve with $r > 1$ branches C_i. Then*

$$d(C) = \inf\{\inf_i\{d(C_i)\}, \inf_{i,j}\{d(C_i, C_j)\}\}.$$

Moreover, there exists $i_0 \in \{1, \dots, r\}$ such that if a smooth branch W has maximal contact with the branch C_{i_0} then it has maximal contact with the curve C.

Proof. Use Theorem 2.3 when $\delta = d$ and $\mathcal{B}$ is the family of smooth branches. □

Corollary 3.4. *The contact exponent of a reduced curve is an equisingularity invariant.*

Proof. It is a consequence of Lemma 3.2 and Proposition 3.3. □

Corollary 3.5. *Let C be a reduced curve with $r \geq 1$ branches. Then $d(C)$ equals ∞ or a rational number greater than or equal to 1. There exists a smooth curve W that has maximal contact with C. Moreover,*

(1) *$d(C) = +\infty$ if and only if C is a smooth branch.*
(2) *$d(C) = 1$ if and only if C has at least two tangents.*
(3) *$d(C) < \inf_{i=1}^{r}\{d(C_i)\}$ if and only if $d(C)$ is an integer.*

Proof. The first and second properties follow from Lemma 3.2 and Proposition 3.3.

To check the third part suppose that $d(C) \in \mathbb{N}$. Then $d(C) \neq \inf_i\{d(C_i)\}$ and by Proposition 3.3 we get the inequality $d(C) < \inf_i\{d(C_i)\}$.

Suppose now that $d(C) < \inf_i\{d(C_i)\}$. We have to check that $d(C) \in \mathbb{N}$. By Proposition 3.3 we get $d(C) = \inf_{i,j}\{d(C_i, C_j)\} = d(C_{i_0}, C_{j_0})$ for some i_0, j_0. By hypothesis $d(C) = d(C_{i_0}, C_{j_0}) < \inf\{d(C_{i_0}), d(C_{j_0})\}$. Hence by Proposition 2.4 ($\delta = d$) there is not a branch with maximal contact with C_{i_0} and C_{j_0} and $d(C) = d(C_{i_0}, C_{j_0}) = d(U, V)$ for some smooth branches U, V, and we conclude that $d(C) \in \mathbb{N}$. □

Lemma 3.6. *Let C and D be two branches with common tangent. Suppose that $m(C) = m(\widehat{C})$ and $m(D) = m(\widehat{D})$. Then*

$$d(C, D) = d(\widehat{C}, \widehat{D}) + 1.$$

Proof. It is a consequence of Max Noether's theorem, which states $(C, D)_0 = m(C)m(D) + (\widehat{C}, \widehat{D})_0$. □

Theorem 3.7 (Hironaka)**.** *Let $\widehat{C}$ be the strict quadratic transformation of a reduced singular unitangent curve C. Then*

(i) *if $d(C) < 2$ then $m(\widehat{C}) < m(C)$,*
(ii) *if $d(C) \geq 2$ then $m(\widehat{C}) = m(C)$ and $d(\widehat{C}) = d(C) - 1$,*
(iii) *if $d(C) \geq 2$ and W is a smooth curve tangent to C then $d(C, W) = d(\widehat{C}, \widehat{W}) + 1$. If W has maximal contact with C then $\widehat{W}$ has maximal contact with $\widehat{C}$.*

Proof. Firstly consider the case when C is a singular branch. Let $\Gamma(C) = \langle v_0, v_1, \ldots, v_g \rangle$. Let us prove (i). By Lemma 3.2 $d(C) = \frac{v_1}{v_0}$ so $d(C) < 2$ if and only if $v_1 - v_0 < v_0$. By the second part of Lemma 1.1 we have $m(\widehat{C}) = \min(\Gamma(\widehat{C}) \backslash \{0\}) = v_1 - v_0 < v_0 = \min(\Gamma(C) \backslash \{0\}) = m(C)$.

Now we will prove (ii) when C is irreducible. Assume that $d(C) \geq 2$ (in fact $d(C) > 2$ since $d(C) \notin \mathbb{N}$). The condition $d(C) \geq 2$ means $v_0 < v_1 - v_0$ and by the first part of Lemma 1.1 we get $\Gamma(\widehat{C}) = \langle v_0, v_1 - v_0, \cdots \rangle$. Consequently $m(\widehat{C}) = v_0 = m(C)$ and $d(\widehat{C}) = \frac{v_1 - v_0}{v_0} = d(C) - 1$.

Now let $C = \bigcup_{i=1}^{r} C_i$, $r > 1$ with irreducible C_i and let us prove (i) and (ii) in this case.
Assume that $d(C) < 2$. We claim that there exists $i_0 \in \{1, \ldots, r\}$ such that $d(C) = d(C_{i_0})$. Suppose that such i_0 does not exist. Then $d(C) \neq d(C_i)$ for any $i \in \{1, \ldots, r\}$ and by Proposition 3.3 $d(C) = \inf_{i,j}\{d(C_i, C_j)\} = d(C_{i_0}, C_{j_0})$ for some $i_0, j_0 \in \{1, \ldots, r\}$. We claim that $d(C_{i_0}) < 2$ or $d(C_{j_0}) < 2$. In the contrary case, we had $d(C_{i_0}) \geq 2$ and $d(C_{j_0}) \geq 2$ and we would get $m(\widehat{C}_{i_0}) = m(C_{i_0})$ and $m(\widehat{C}_{j_0}) = m(C_{j_0})$, which implies by Lemma 3.6 $d(C_{i_0}, C_{j_0}) = d(\widehat{C}_{i_0}, \widehat{C}_{j_0}) + 1 \geq 2$. This is a contradiction since $d(C_{i_0}, C_{j_0}) = d(C) < 2$.
If $d(C_{i_0}) = d(C) < 2$ then by the irreducibility case, $m(\widehat{C}_{i_0}) < m(C_{i_0})$ and $m(C) - m(\widehat{C}) = \sum_{i=1}^{r}(m(C_i) - m(\widehat{C}_i)) \geq m(C_{i_0}) - m(\widehat{C}_{i_0}) > 0$.

Suppose now that $d(C) \geq 2$. We have

$$\inf\{d(C_i)\} \geq \inf\{\inf(d(C_i)), \inf(d(C_i, C_j))\} = d(C) \geq 2.$$

Thus $d(C_i) \geq 2$ for $i \in \{1, \ldots, r\}$ and by the first part of the proof $m(\widehat{C}_i) = m(C_i)$ and $d(\widehat{C}_i) = d(C_i) - 1$. Hence $m(\widehat{C}) = m(C)$. Moreover, by Lemma 3.6, $d(C_i, C_j) = d(\widehat{C}_i, \widehat{C}_j) + 1$ and $d(C) = \inf\{\inf(d(C_i)), \inf(d(C_i, C_j))\} = \inf\{\inf(d(\widehat{C}_i)), \inf(d(\widehat{C}_i, \widehat{C}_j))\} + 1 = d(\widehat{C}) + 1$.

To finish let us prove (iii). By Lemma 3.6 $d(C_i, W) = d(\widehat{C}_i, \widehat{W}) + 1$ for $i \in \{1, \ldots, r\}$ and $d(C, W) = \inf\{d(C_i, W)\} = \inf\{d(\widehat{C}_i, W)\} + 1 = d(\widehat{C}, W) + 1$. Suppose that W has maximal contact with C. Then $d(C) = d(C, W) = d(\widehat{C}, \widehat{W}) + 1 \leq d(\widehat{C}) + 1 = d(C)$, where the last equality is a consequence of statement (ii) of the theorem. This implies $d(\widehat{C}, \widehat{W}) = d(\widehat{C})$. Thus $\widehat{W}$ has maximal contact with $\widehat{C}$. □

Lemma 3.8. *Let C be a reduced curve with $r > 1$ branches and W a smooth branch. If $d(C, W) \notin \mathbb{N}$ then $d(C, W) = d(C)$.*

Proof. The lemma is obvious if C is a branch. In the general case $d(C, W) = \inf_i\{d(C_i, W)\} = d(C_{i_0}, W)$ for some $i_0 \in \{1, \ldots, r\}$. If $d(C, W) \notin \mathbb{N}$ then $d(C_{i_0}, W) \notin \mathbb{N}$ and $d(C_{i_0}, W) = d(C_{i_0})$ since C_{i_0} is a branch. Consequently, we get $d(C, W) = d(C_{i_0})$ which implies, by Proposition 3.3, $d(C) = d(C_{i_0}) = d(C, W)$. □

Now we give a characterization of smooth curves which does not have maximal contact with a reduced curve.

Proposition 3.9. *Let C be a reduced curve with $r > 1$ branches. A smooth branch W does not have maximal contact with C if and only if $(C,W)_0 < d(C)m(C)$. Moreover, in this case $(C,W)_0 \equiv 0 \pmod{m(C)}$.*

Proof. Let us suppose that W is a smooth branch which does not have maximal contact with C. We will check that $(C,W)_0 < d(C)m(C)$ and $\frac{(C,W)_0}{m(C)} \in \mathbb{N}$. By Proposition 3.3 we get $d(C,W) < \inf_{i,j}\{d(C_i,C_j)\}$ since $d(C,W) < d(C)$. According to the second part of Lemma 3.1 we can write $d(C,W) = \frac{(C,W)_0}{m(C)}$, thus $(C,W)_0 = d(C,W)m(C) < d(C)m(C)$. We claim $\frac{(C,W)_0}{m(C)} = d(C,W)$ is an integer. Indeed, by Lemma 3.8 we get $d(C,W) = d(C)$, which is a contradiction.

Now suppose that $(C,W)_0 < d(C)m(C)$. By the second part of Lemma 3.1 we get $d(C,W) \le \frac{(C,W)_0}{m(C)}$ and consequently $d(C,W) < d(C)$, which means that W does not have maximal contact with C. □

4. Milnor number and Hironaka contact exponent

Let C be a reduced curve. We define the *Milnor number* $\mu(C)$ of C by the formula $\mu(C) = c(C) - r(C) + 1$, where $c(C)$ is the degree of the conductor of the local ring of C and $r(C)$ is the number of branches (see Preliminaries).

If $C : \{f = 0\}$ then $\mu(C) = \dim_K K[[x,y]]/\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$ provided that K is of characteristic zero (see [GB-P2]).

Lemma 4.1. *Let $C = \bigcup_{i=1}^r C_i$, where $r \ge 1$ and C_i are irreducible. Then*

(1) $\mu(C) + r - 1 = \sum_{i=1}^r \mu(C_i) + 2\sum_{1\le i<j\le r}(C_i,C_j)_0$,
(2) *if C is a branch then $\mu(C)$ equals the conductor of the semigroup $\Gamma(C)$,*
(3) *$\mu(C) \ge 0$ with equality if and only if C is a smooth branch.*

Proof. See [GB-P2, Proposition 2.1]. □

Proposition 4.2. *Let $C = \bigcup_{i=1}^r C_i$ be a singular reduced curve with r branches C_i. Then $\mu(C) \ge (d(C)m(C) - 1)(m(C) - 1)$ with equality if and only if the following two conditions are satisfied:*

(e_1) *$d(C_i,C_j) = d(C)$ for all $i \ne j$,*
(e_2) *if the branch C_i is singular then C_i has exactly one Zariski pair and $d(C_i) = d(C)$.*

Proof. First let us suppose that C is a branch with $\Gamma(f) = \langle v_0, v_1, \ldots, v_g\rangle$. Let $n_0 = 1$. Since $n_{i-1}v_{i-1} \le v_i$ for $i \in \{1,\ldots,g\}$ we have $n_0n_1\cdots n_{k-1}v_1 \le v_k$ for

$k \in \{1, \dots, g\}$. We get

$$\begin{aligned} c(C) &= \sum_{k=1}^{g}(n_k - 1)v_k - v_0 + 1 \geq \sum_{k=1}^{g}((n_k - 1)n_{k-1}\cdots n_1 n_0)v_1 - v_0 + 1 \\ &= (n_g \cdots n_1 n_0 - n_0)v_1 - (v_0 - 1) = (v_0 - 1)v_1 - (v_0 - 1) \\ &= (v_0 - 1)(v_1 - 1). \end{aligned}$$

Moreover, $c(C) = (v_0 - 1)(v_1 - 1)$ if and only if $\Gamma(C) = \langle v_0, v_1 \rangle$.

Now suppose that the curve C has $r > 1$ branches C_i and let $\overline{m}_i = m(C_i)$ for $i \in \{1, \dots, r\}$. From Proposition 3.3 we get $d(C_i) \geq d(C)$ and $d(C_i, C_j) \geq d(C)$ for all $i, j \in \{1, \dots, r\}$. By the first part of the proof $\mu(C_i) \geq (d(C_i)\overline{m}_i - 1)(\overline{m}_i - 1)$ for the singular branches with equality if and only if C_i is a singular branch satisfying condition (e_2).

Let $I := \{i \;:\; C_i \text{ is singular}\}$. Now we get

$$\begin{aligned} \mu(C) + r - 1 &= \sum_{i=1}^{r}\mu(C_i) + 2\sum_{1\leq i<j\leq r}(C_i, C_j)_0 \\ &= \sum_{i=1}^{r}\mu(C_i) + 2\sum_{1\leq i<j\leq r} d(C_i, C_j)\overline{m}_i\overline{m}_j \\ &\geq \sum_{i\in I}(d(C_i)\overline{m}_i - 1)(\overline{m}_i - 1) + 2\sum_{1\leq i<j\leq r} d(C_i, C_j)\overline{m}_i\overline{m}_j \\ &\geq \sum_{i=1}^{r}(d(C)\overline{m}_i - 1)(\overline{m}_i - 1) + 2\sum_{1\leq i<j\leq r} d(C)\overline{m}_i\overline{m}_j \\ &= d(C)(m(C)^2 - m(C)) - m(C) + r \end{aligned}$$

with equality if and only if the conditions (e_1) and (e_2) are satisfied. $\square$

Lemma 4.3. *Let C be a unitangent singular curve. We have:*

(1) $d(C) \geq 1 + \frac{1}{m(C)}$. *Moreover* $d(C) = 1 + \frac{1}{m(C)}$ *if and only if C is a branch with semigroup* $\langle m(C), m(C) + 1 \rangle$.
(2) $\mu(C) \geq m(C)(m(C) - 1)$ *with equality if and only if* $d(C) = 1 + \frac{1}{m(C)}$.

Proof. Let $\{C_i\}_i$ be the set of branches of C. To check the first part of the lemma we may assume that $d(C)$ is not an integer. Then by Proposition 3.3 and the third part of Corollary 3.5 there is an i_0 such that $d(C) = d(C_{i_0})$. The contact exponent $d(C_{i_0})$ is a fraction with the denominator less than or equal to $m(C_{i_0})$. Therefore we get $d(C) = d(C_{i_0}) \geq 1 + \frac{1}{m(C_{i_0})} \geq 1 + \frac{1}{m(C)}$ and the equality $d(C) = 1 + \frac{1}{m(C)}$ implies $m(C_{i_0}) = m(C)$ and consequently $C_{i_0} = C$. Moreover the semigroup of C is $\langle m(C), m(C) + 1 \rangle$ since $m(C)$ and $m(C) + 1$ are coprime.

In order to prove the second part we get, by Proposition 4.2 and the first part of this lemma,

$$\begin{aligned}\mu(C) &\geq (d(C)m(C)-1)(m(C)-1)\\ &\geq \left(\left(1+\frac{1}{m(C)}\right)m(C)-1\right)(m(C)-1)=m(C)(m(C)-1).\end{aligned}$$

If $\mu(C)=m(C)(m(C)-1)$ then from the above calculation it follows that $d(C)=1+\frac{1}{m(C)}$.

On the other hand if $d(C)=1+\frac{1}{m(C)}$ then by the first part of this lemma C is a branch of semigroup $\langle m(C), m(C)+1\rangle$. According to Proposition 4.2 $\mu(C)=(d(C)m(C)-1)(m(C)-1)=m(C)(m(C)-1)$. □

If $\mu(C)=(d(C)m(C)-1)(m(C)-1)$ then the pair $(m(C),d(C))$ determines the equisingularity class of C. More specifically, we have:

Proposition 4.4. *Let C be a reduced singular curve. Then $\mu(C)=(d(C)m(C)-1)(m(C)-1)$ if and only if one of the following three conditions holds*

1. $d(C)\in\mathbb{N}$. *All branches of C are smooth and intersect pairwise with multiplicity $d(C)$.*
2. $d(C)\notin\mathbb{N}$ *and* $m(C)d(C)\in\mathbb{N}$. *The curve C has* $r=\gcd(m(C),m(C)d(C))$ *branches, each with semigroup generated by* $\left(\frac{m(C)}{r},\frac{m(C)d(C)}{r}\right)$*, intersecting pairwise with multiplicity* $\frac{m(C)^2d(C)}{r^2}$.
3. $m(C)d(C)\notin\mathbb{N}$. *There is a smooth curve L such that $C=L\cup C'$, where C' is a curve of type (2) with $d(C')=d(C)$ and $m(C')=m(C)-1$. The branch L has maximal contact with any branch of C'.*

Proof. If one of conditions (1), (2) or (3) is satisfied then a direct calculation shows that $\mu(C)=(d(C)m(C)-1)(m(C)-1)$.

Suppose that $C=\bigcup_{i=1}^r C_i$ satisfy the equality $\mu(C)=(d(C)m(C)-1)(m(C)-1)$. By Proposition 4.2 the conditions (e_1) and (e_2) are satisfied. Let us consider three cases:

Case 1: All branches C_i are smooth. Then C is of type (1) by (e_1).

Case 2: All branches C_i are singular. Then the branches $\{C_i\}_i$ have the same semigroup $\langle v_0,v_1\rangle$ and according to (e_2) $d(C_i)=d(C)$ for all $i\in\{1,\dots,r\}$. Clearly, we have $m(C)=\sum_{i=1}^r m(C_i)=rv_0$ and $m(C)d(C)=m(C)d(C_i)=rv_1$. Thus $m(C)d(C)\in\mathbb{N}$, $r=\gcd(m(C),m(C)d(C))$ and it is easy to see that C is of type (2).

Case 3: Neither Case 1 nor Case 2 holds, thus $r>1$. We may assume that C_1 is smooth and C_2 is singular. If $r>2$ then all branches C_i for $i\geq 3$ are singular. In fact, we have by (e_1): $d(C_1,C_i)=d(C_2,C_i)=d(C_1,C_2)=d(C)$ and by (e_2):

$d(C) = d(C_2) \notin \mathbb{N}$. Thus $d(C_1, C_i) \notin \mathbb{N}$ and C_i are singular for all $i \geq 3$. Let $L := C_1$ and $C' := \bigcup_{i=2}^{r}(C_i, 0)$. Then $C = L \cup C'$ and we check using Proposition 3.3 that C is of type (3). $\square$

Corollary 4.5. *Let C_1, C_2 be two reduced singular curves such that $\mu(C_i) = (d(C_i)m(C_i) - 1)(m(C_i) - 1)$ for $i \in \{1, 2\}$. Then C_1 and C_2 are equisingular if and only if $(m(C_1), d(C_1)) = (m(C_2), d(C_2))$.*

Corollary 4.6. *Let C be a reduced singular curve. Suppose that $\mu(C) = (d(C)m(C) - 1)(m(C) - 1)$ and $m(C)d(C) \notin \mathbb{N}$. Then $(m(C) - 1)d(C) \in \mathbb{N}$.*

To compute $\mu(C)$ one can use Pham's formula.

Proposition 4.7 ([Ph])**.** *Let $C = \bigcup_i^t C_i$, where C_i are unitangent and the tangents to C_i and C_j are different for $i \neq j$. Then*

$$\mu(C) + t - 1 = m(C)(m(C) - 1) + \sum_{k=1}^{t} \mu(\widehat{C}_k).$$

Proof. We distinguish three cases.

Suppose that C is irreducible. Then $\mu(C) = m(C)(m(C) - 1) + \mu(\widehat{C})$ by the well-known formula $c(C) = m(C)(m(C) - 1) + c(\widehat{C})$ (see [Ang, Korollar II.1.8]).

Suppose now that C is unitangent and let $C = \bigcup_{i=1}^{r} C_i$, where C_i are irreducible and let $\overline{m}_i = m(C_i)$ for $i \in \{1, \ldots, r\}$. Then

$$\begin{aligned}
\mu(C) + r - 1 &= \sum_{i=1}^{r} \mu(C_i) + 2 \sum_{1 \leq i < j \leq r} (C_i, C_j)_0 \\
&= \sum_{i=1}^{t} \Big(\overline{m}_i(\overline{m}_i - 1) + \mu(\widehat{C_i})\Big) + 2 \sum_{1 \leq i < j \leq r} \Big(\overline{m}_i \overline{m}_j + (\widehat{C_i}, \widehat{C_j})_0\Big) \\
&= \sum_{i=1}^{r} \overline{m}_i(\overline{m}_i - 1) + 2 \sum_{1 \leq i < j \leq r} \overline{m}_i \overline{m}_j + \sum_{i=1}^{r} \mu(\widehat{C_i}) \\
&\qquad + 2 \sum_{1 \leq i < j \leq r} (\widehat{C_i}, \widehat{C_j})_0 \\
&= \sum_{i=1}^{r} \overline{m}_i(\overline{m}_i - 1) + 2 \sum_{1 \leq i < j \leq r} \overline{m}_i \overline{m}_j + \mu(\cup_{i=1}^{r} \widehat{C_i}) + r - 1 \\
&= \sum_{i=1}^{r} \overline{m}_i(\overline{m}_i - 1) + 2 \sum_{1 \leq i < j \leq r} \overline{m}_i \overline{m}_j + \mu(\widehat{\cup_{i=1}^{r} C_i}) \\
&= m(C)(m(C) - 1) + \mu(\widehat{C}) + r - 1.
\end{aligned}$$

Finally suppose that $C=\bigcup_{i=1}^{t} C_i$, where C_i are unitangent and the tangents to C_i and C_j are different for $i \neq j$. Put $\overline{m}_i = m(C_i)$ for $i \in \{1,\dots,t\}$. Then

$$\begin{aligned}
\mu(C)+t-1 &= \sum_{i=1}^{t}\mu(C_i) + 2\sum_{1\le i<j\le t}(C_i,C_j)_0 \\
&= \sum_{i=1}^{t}\left(\overline{m}_i(\overline{m}_i-1)+\mu(\widehat{C_i})\right) + 2\sum_{1\le i<j\le t}\overline{m}_i\overline{m}_j \\
&= \sum_{i=1}^{t}\mu(\widehat{C_i}) + \sum_{i=1}^{t}(\overline{m}_i)^2 + 2\sum_{1\le i<j\le t}\overline{m}_i\overline{m}_j - \sum_{i=1}^{t}\overline{m}_i \\
&= m(C)(m(C)-1) + \sum_{i=1}^{t}\mu(\widehat{C_i}).
\end{aligned}$$

□

5. Contact exponents of higher order

Let $\mathcal{B}_k$ be the family of branches having at most $k-1$ Zariski pairs. If C is a reduced curve we put

$$d_k(C) := \sup\{d(C,W)\ :\ W \in \mathcal{B}_k\} = d(C,\mathcal{B}_k).$$

Observe that $d_1(C)=d(C)$.

A branch $D \in \mathcal{B}_k$ has k-maximal contact with C if $d(C,D)=d_k(C)$.

The concept of contact exponent of higher order was studied by Lejeune-Jalabert [LJ] and Campillo [C].

Lemma 5.1. *Let $C : \{f=0\}$ be a singular branch with $\Gamma(C)=\langle v_0,v_1,\dots,v_g\rangle$. There exist irreducible power series $f_0,\dots,f_{g-1}$ such that* $\operatorname{ord} f_{k-1} = \frac{v_0}{e_{k-1}}$ *and $(f,f_{k-1})_0 = v_k$.*

Proof. We may assume that $(f,x)_0 = \operatorname{ord} f$. According to [GB-P1, Theorem 3.2] there exist distinguished polynomials $f_0,\dots,f_{g-1}$ such that $(f_{k-1},x)_0 = \frac{v_0}{e_{k-1}}$ and $(f,f_{k-1})_0 = v_k$. Consider the log-distances $d(f,x)=1$, $d(f_{k-1},x) = \frac{(f_{k-1},x)_0}{\operatorname{ord} f_{k-1}}$ and $d(f,f_k) = \frac{v_k}{v_0\frac{v_0}{e_{k-1}}} = \frac{e_{k-1}v_k}{(v_0)^2}$. Since $d(f_{k-1},x) = \frac{e_{k-1}v_k}{(v_0)^2} \ge \frac{e_0 v_1}{(v_0)^2} = \frac{v_1}{v_0} > 1$ we have $d(f_{k-1},x)=d(f,x)=1$, that is $(f_{k-1},x)_0 = \operatorname{ord} f_{k-1}$. □

Lemma 5.2. *Let $C : \{f=0\}$ be a singular branch with $\Gamma(C)=\langle v_0,v_1,\dots,v_g\rangle$. If E is a branch such that $d(C,E) > \frac{e_{k-1}v_k}{(v_0)^2}$ then E has at least k Zariski pairs.*

Proof. See [GB-P1, Theorem 5.2]. □

Proposition 5.3. *Let C be a branch with $\Gamma(C)=\langle v_0,\dots,v_g\rangle$. Then $d_k(C) = \frac{e_{k-1}v_k}{(v_0)^2}$.*

Proof. By Lemma 5.1 there is $D_{k-1} \in \mathcal{B}_k$ such that ord $D_{k-1} = \frac{v_0}{e_{k-1}}$ and $(C, D_{k-1})_0 = v_k$. Then $d_k(C) \geq d(C, D_{k-1}) = \frac{(C,D_{k-1})_0}{\text{ord } C \text{ ord } D_{k-1}} = \frac{e_{k-1}v_k}{(v_0)^2}$. Suppose now that there is a branch $E \in \mathcal{B}_k$ such that $d(C,E) > \frac{e_{k-1}v_k}{(v_0)^2}$. Then $\frac{(C,E)_0}{v_0 \text{ord } E} > \frac{e_{k-1}v_k}{(v_0)^2}$, hence $\frac{(C,E)_0}{\text{ord } E} > \frac{e_{k-1}v_k}{v_0}$. By Lemma 5.2 we conclude that E has at least k Zariski pairs which is a contradiction (since $E \in \mathcal{B}_k$). □

Proposition 5.4. *Let C be a reduced curve with $r > 1$ branches C_i. Then*

$$d_k(C) = \inf\{\inf_i\{d_k(C_i)\}, \inf_{i,j}\{d(C_i, C_j)\}\}.$$

Proof. Use Theorem 2.3 when $\delta = d$ and $\mathcal{B}_k$ is the family of branches having at most $k-1$ Zariski pairs. □

6. Polar invariants and the contact exponent

Let K be a field of characteristic zero. Let C be a reduced plane singular curve and let $P(C)$ be a generic polar of C. Then $P(C)$ is a reduced germ of multiplicity $m(P(C)) = m(C) - 1$. Let $P(C) = \bigcup_{j=1}^{s} D_j$ be the decomposition of $P(C)$ into branches D_j.

We put $\mathcal{Q}(C) = \left\{ \frac{(C,D_j)_0}{m(D_j)} \;:\; j \in \{1, \ldots, s\} \right\}$ and call the elements of $\mathcal{Q}(C)$ the *polar invariants* of C. They are equisingularity invariants of C (see [T2], [Gw-P]). In particular if C is a branch then

$$\mathcal{Q}(C) := \{m(C)d_k(C)\}_{k=1}^{g}.$$

Let us consider the minimal polar invariant $\alpha(C) := \inf \mathcal{Q}(C)$.

Proposition 6.1. *For any singular reduced germ C we have $\alpha(C) = m(C)d(C)$.*

Proof.- See [L-M-P, Theorem 2.1 (iii)].

One could prove Proposition 6.1 by using Theorem 3.3 and the explicit formulae for the polar invariants given in [Gw-P, Theorem 1.3].

We say that C is an *Eggers singularity* if $\mathcal{Q}(C)$ has exactly one element.

Proposition 6.2. *Let C be a singular reduced curve. Then $\mu(C) = (d(C)m(C) - 1)(m(C) - 1)$ if and only if C is an Eggers singularity.*

Proof. By [T1] Proposition 1.2 we get

$$\begin{aligned}
\mu(C) &= (C, P(C))_0 - m(C) + 1 = \sum_{j=1}^{s}(C, D_j)_0 - m(C) + 1 \\
&\geq \alpha(C)m(P(C)) - m(C) + 1 = \alpha(C)(m(C) - 1) - m(C) + 1 \\
&= (\alpha(C) - 1)(m(C) - 1)
\end{aligned}$$

with equality if and only if C is an Eggers singularity. We use Proposition 6.1. $\square$

Proposition 4.4 provides an explicit description of Eggers singularities.

Corollary 6.3. *(*[E, p. 16]*) If C has exactly one polar invariant then C is equisingular to $y^n - x^m = 0$ or $y^n - yx^m = 0$, for some integers $1 < n < m$.*

Proof. We check that if $C : \{y^n - x^m = 0\}$ then $m(C) = n$, $d(C) = \frac{m}{n}$ and $\mu(C) = nm - n - m + 1$. On the other hand if $C : \{y^n - yx^m = 0\}$ then $m(C) = n$, $d(C) = \frac{m}{n-1}$ and $\mu(C) = nm - n + 1$. In both cases $\mu(C) = (d(C)m(C) - 1)(m(C) - 1)$, that is C is an Eggers singularity.

Now let C be an Eggers singularity. If $m(C)d(C) \in \mathbb{N}$ then C and $\{y^{m(C)} - x^{m(C)d(C)} = 0\}$ are equisingular by Corollary 4.5. Analogously, if $m(C)d(C) \notin \mathbb{N}$ then, by Corollary 4.6, $(m(C) - 1)d(C) \in \mathbb{N}$ and C is equisingular to $\{y^{m(C)} - yx^{(m(C)-1)d(C)} = 0\}$. $\square$

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