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## POLAR INVARIANTS OF PLANE CURVE SINGULARITIES: INTERSECTION THEORETICAL APPROACH

*Dedicated to Professor Pierrette Cassou-Noguès*

**Abstract.** This article, based on the talk given by one of the authors at the Pierrettest in Castro Urdiales in June 2008, is an overview of a number of recent results on the polar invariants of plane curve singularities.

### Introduction

The polar invariants (called also polar quotients) of isolated hypersurface singularities were introduced by B. Teissier in 1975 to study equisingularity problems (see [Te1975], [Te1977], [Te1980]). They are by definition, the contact orders between a hypersurface and the branches of its generic polar curve. To every polar invariant  $q$  of a given isolated hypersurface singularity one associates in a natural way an integer  $m_q > 0$  called the multiplicity of  $q$ . Teissier's collection  $\{(q, m_q)\}$  is an analytic invariant of the singularity. Even more: it is an invariant of the "c-cosécance" which is equivalent in the case of plane curve singularities to the constancy of the local embedded topological type (see [Te1977]). The Milnor number, the Łojasiewicz exponent, the  $C^0$ -degree of sufficiency and other numerical invariants can be computed in terms of Teissier's collection.

It is well-known (see [Te1976], [BriKn1986], [Te1991]) that the constancy of the local embedded topological type of plane curves is equivalent to the usual definitions of equisingularity (see Preliminaries where the definition of equisingularity in terms of intersection numbers is given).

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M. Merle [Mer1977] computed Teissier's collection for a branch (irreducible analytic curve) in terms of the semigroup of the branch. Much earlier a computation of the contacts between an irreducible curve and the branches of its generic polar curve was done by Henry J. S. Smith [Sm1875] but his work fell into oblivion for long time. R. Ephraim [Eph1983] generalized the Smith-Merle result computing the polar invariants in the case of special polars and applied his result to the pencil of curves which appears when studying affine curves with one branch at infinity (see Sections 4 and 7 of this article).

The case of multi-branched curves turned out much more complicated and was studied by many authors: Eggers [Egg1982], Delgado [Del1994], Casas-Alvero [Cas2000], García Barroso [Gar2000], C. T. C. Wall [Wall2003] using algebraic methods and by Lê D. T., F. Michel and C. Weber in [LêMiWe1989], [LêMiWe1991] using topological tools. Lê D. T. initiated the topological approach to the polar invariants in [Lê1975].

C. T. C. Wall gave an account of most results obtained in the above quoted papers in his book [Wall2004] dealing with different aspects of the curve singularities.

The goal of this article is to give an overview of a number of recent results on the polar invariants of plane curve singularities.

In Section 2 we present a refinement of Teissier's invariance theorem in the case of plane curve singularities. In Section 3 we give an approach to the polar invariants based on Puiseux series developing the method due to Kuo and Lu [KuoLu1977].

Section 4 is devoted to the Smith-Merle-Ephraim theorem in the one branch case and to the irreducibility criterion obtained quite recently by García Barroso and Gwoździewicz. (Theorem 4.5 and Corollary 4.6).

In Section 5 we present explicit formulae for the polar invariants in terms of semigroup of branches and intersection multiplicities due to Gwoździewicz and Płoski (Theorem 5.2). The geometric interpretation of these formulae in terms of the Newton diagrams associated with many-branched singularity is new.

In Section 6 we recall a result obtained by Lenarcik and Płoski (Theorem 6.1) which gives an effective formula for the jacobian Newton diagram (see Section 2) of a nondegenerate (in the sense of Kouchnirenko) plane curve singularity. Then, we present in Section 7, some applications of the polar invariants to pencils of plane curve singularities.

## 1. Preliminaries

In this section we recall some useful notions and results that we need in this article. The references for this part are [BriKn1986], [Cas2000], [Te1991], [Wall2004].

### 1.1. Basic notions

Let  $\mathbf{C}\{X, Y\}$  be the ring of convergent complex power series in variables  $X, Y$ . Let  $f \in \mathbf{C}\{X, Y\}$  be a nonzero power series without constant term. An *analytic curve*  $f = 0$  is defined to be the ideal generated by  $f$  in  $\mathbf{C}\{X, Y\}$ . We say that  $f = 0$  is *irreducible (reduced)* if  $f \in \mathbf{C}\{X, Y\}$  is irreducible ( $f$  has no multiple factors). The irreducible curves are also called branches. If  $f = f_1^{m_1} \dots f_r^{m_r}$  with non-associated irreducible factors  $f_i$  then we refer to  $f_i = 0$  as the branches or components of  $f = 0$ .

Recall here that for any nonzero power series  $f = \sum c_{\alpha\beta} X^\alpha Y^\beta$  we put  $\text{ord } f = \inf\{\alpha + \beta : c_{\alpha\beta} \neq 0\}$  and in  $f = \sum c_{\alpha\beta} X^\alpha Y^\beta$  with summation over  $(\alpha, \beta)$  such that  $\alpha + \beta = \text{ord } f$ . The initial form in  $f$  of  $f$  determines the *tangents to*  $f = 0$ .

For any power series  $f, g \in \mathbf{C}\{X, Y\}$  we define the *intersection number*  $(f, g)_0$  by putting

$$(f, g)_0 = \dim_{\mathbf{C}} \mathbf{C}\{X, Y\} / (f, g)$$

where  $(f, g)$  is the ideal of  $\mathbf{C}\{X, Y\}$  generated by  $f$  and  $g$ . If  $f, g$  are nonzero power series without constant term then  $(f, g)_0 < +\infty$  if and only if the curves  $f = 0$  and  $g = 0$  have no common branch.

Now suppose that  $f = 0$  is a branch and consider

$$S(f) = \{(f, g)_0 : g \in \mathbf{C}\{X, Y\} \text{ runs over all series such that } f \text{ does not divide } g\}.$$

Clearly  $0 \in S(f)$  (take  $g = 1$ ) and  $a, b \in S(f) \Rightarrow a + b \in S(f)$  since the intersection number is additive. We call  $S(f)$  the *semigroup of the branch*  $f = 0$ . Note that  $S(f) = \mathbf{N}$  if and only if  $\text{ord } f = 1$  (we say then that  $f = 0$  is *regular* or *nonsingular*).

Consider two reduced curves  $f = 0$  and  $g = 0$ . They are *equisingular* if and only if there are factorizations  $f = \prod_{i=1}^r f_i$  and  $g = \prod_{i=1}^r g_i$  with the same number  $r > 0$  of irreducible factors  $f_i$  and  $g_i$  such that

- $S(f_i) = S(g_i)$  for all  $i = 1, \dots, r$ ,
- $(f_i, f_j)_0 = (g_i, g_j)_0$  for  $i, j = 1, \dots, r$ .

The bijection  $f_i \mapsto g_i$  will be called *equisingularity bijection*. In particular two branches are equisingular if and only if they have the same semigroup. A function defined on the set of reduced curves is an *invariant* if it is constant on equisingular curves. The multiplicity  $\text{ord } f$ , the number of branches  $r(f)$  and the number of tangents  $t(f)$  of  $f = 0$  are invariants.

For any analytic curve  $f = 0$  we consider the *Milnor number*  $\mu_0(f) = (\partial f / \partial X, \partial f / \partial Y)_0$ . One has  $\mu_0(f) < +\infty$  if and only if the curve  $f = 0$  is reduced. Let us recall the following two properties:

- if  $f = 0$  is a branch then  $\mu_0(f)$  is the smallest integer  $c \geq 0$  such that all integers greater than or equal to  $c$  belong to  $S(f)$ ,
- if  $f = f_1 \dots f_r$  with pairwise different irreducible  $f_i$  then

$$\mu_0(f) + r - 1 = \sum_{i=1}^r \mu_0(f_i) + 2 \sum_{1 \leq i < j \leq r} (f_i, f_j)_0.$$

Thus the Milnor number is an invariant. A simple proof of the above properties is given in [P1995].

**1.2. Newton diagrams after [Te1976]**

Let  $\mathbf{R}_+ = \{x \in \mathbf{R} : x \geq 0\}$ . The Newton diagrams are some convex subsets of  $\mathbf{R}_+^2$ . Let  $E \subset \mathbf{N}^2$  and let us denote by  $\Delta(E)$  the convex hull of the set  $E + \mathbf{R}_+^2$ . The subset  $\Delta$  of  $\mathbf{R}_+^2$  is a *Newton diagram* (or *polygon*) if there is a set  $E \subset \mathbf{N}^2$  such that  $\Delta = \Delta(E)$ . The smallest set  $E_0 \subset \mathbf{N}^2$  such that  $\Delta = \Delta(E_0)$  is called the *set of vertices* of the Newton diagram  $\Delta$ . It is always finite and we can write  $E_0 = \{v_0, v_1, \dots, v_m\}$  where  $v_i = (\alpha_i, \beta_i)$  and  $\alpha_{i-1} < \alpha_i, \beta_{i-1} > \beta_i$  for all  $i = 1, \dots, m$ . In particular the Newton diagram  $\Delta$  with one vertex  $v = (\alpha, \beta)$  is the quadrant  $(\alpha, \beta) + \mathbf{R}_+^2$ .

According to Teissier for  $k, l > 0$  we denote by  $\{\frac{k}{l}\}$  the Newton diagram with vertices  $(0, l)$  and  $(k, 0)$ . We put also  $\{\frac{k}{\infty}\} = (k, 0) + \mathbf{R}_+^2$  and  $\{\frac{\infty}{l}\} = (0, l) + \mathbf{R}_+^2$  and call any diagram of the form  $\{\frac{k}{l}\}$  an *elementary Newton diagram*. For any subsets  $\Delta, \Delta' \subset \mathbf{R}_+^2$  we consider the *Minkowski sum*  $\Delta + \Delta' = \{u + v : u \in \Delta \text{ and } v \in \Delta'\}$ . One checks the following

**PROPERTY 1.1.** *The Newton diagrams form the semigroup with respect to the Minkowski sum. The elementary Newton diagrams generate the semigroup of the Newton diagrams.*

For any Newton diagram  $\Delta$  we consider the set  $\mathcal{N}(\Delta)$  of the compact faces of the boundary of  $\Delta$ . If  $\Delta$  has vertices  $v_0, v_1, \dots, v_m$  then  $\mathcal{N}(\Delta) = \{[v_{i-1}, v_i] : i = 1, \dots, m\}$ . For any segment  $S \in \mathcal{N}(\Delta)$  we denote by  $|S|_1$  and  $|S|_2$  the lengths of the projections of  $S$  on the horizontal and vertical axes. We call  $|S|_1/|S|_2$  the *inclination* of  $S$ . If  $\Delta$  intersects both axes then  $\Delta = \sum_S \left\{ \frac{|S|_1}{|S|_2} \right\}$  (summation over all  $S \in \mathcal{N}(\Delta)$ ) and this representation is unique.

Now, let  $f = \sum c_{\alpha\beta} X^\alpha Y^\beta$  be a power series. We put  $\text{supp } f = \{(\alpha, \beta) \in \mathbf{N}^2 : c_{\alpha\beta} \neq 0\}$ ,  $\Delta_{X,Y}(f) = \Delta(\text{supp } f)$  and  $\mathcal{N}_f = \mathcal{N}(\Delta(f))$ . We call  $\Delta_{X,Y}(f)$  the *Newton diagram (or polygon) of the power series f*. Let  $n > 0$  be an integer. Let  $f = f(X, Y)$  be a power series  $Y$ -regular of order  $n$ , i.e. such that  $\text{ord } f(0, Y) = n$ . Let  $\mathbf{C}\{X\}^* = \bigcup_{p \geq 1} \mathbf{C}\{X^{1/p}\}$  be the ring of Puiseux

series. We have the Newton-Puiseux factorization

$$f(X, Y) = U(X, Y) \prod_{i=1}^n (Y - \alpha_i(X)), \quad U(X, Y) \text{ is a unit in } \mathbf{C}\{X, Y\}$$

where  $\alpha_i(X) \in \mathbf{C}\{X\}^*$  for  $i = 1, \dots, n$ .

**THEOREM 1.2.** (Newton-Puiseux Theorem) *For every  $q \in \mathbf{Q} \cup \{\infty\}$  let  $m_q$  be the number of roots  $\alpha_i(X)$  such that  $\text{ord } \alpha_i(X) = q$ . Then  $m_q q$  (by convention  $0 \cdot \infty = 0$ ) is an integer or  $\infty$  and*

$$\Delta_{X,Y}(f) = \sum_q \left\{ \frac{m_q q}{m_q} \right\}.$$

### 1.3. Nondegeneracy

Now, let  $f = \sum c_{\alpha\beta} X^\alpha Y^\beta$  be a power series. For any segment  $S \in \mathcal{N}(f)$  we put  $\text{in}(f, S) = \sum c_{\alpha\beta} X^\alpha Y^\beta$  where  $(\alpha, \beta) \in S$ .

According to [Kou1976], the series  $f$  is *nondegenerate* if for every  $S \in \mathcal{N}(f)$  the polynomial  $\text{in}(f, S)$  has no critical points in the set  $\mathbf{C}^* \times \mathbf{C}^*$ , where  $\mathbf{C}^* = \mathbf{C} \setminus \{0\}$ . A lot of applications of the Newton diagrams are based on the following

**THEOREM 1.3.** ([GarLenPi2007], [Len2008]) *Suppose that  $f, g \in \mathbf{C}\{X, Y\}$  are reduced power series such that  $\Delta(f) = \Delta(g)$ . Then*

- (i) *if  $f$  and  $g$  are nondegenerate then the curves  $f = 0$  and  $g = 0$  are equisingular,*
- (ii) *if  $f$  is nondegenerate but  $g$  is degenerate then  $f = 0$  and  $g = 0$  are not equisingular.*

Let  $\Delta \subset \mathbf{R}_+^2$  be a Newton diagram. It is easy to check that  $\Delta = \Delta(f)$  for a reduced nondegenerate power series  $f$  if and only if the distances from  $\Delta$  to the axes are  $\leq 1$ . We call such diagrams *nearly convenient*. Every Newton diagram which intersects both axes (convenient in the sense of Kouchnirenko) is nearly convenient. If  $\Delta$  is nearly convenient then the reduced nondegenerate power series  $f$  such that  $\Delta = \Delta(f)$  form an open dense subset in the space of coefficients.

Let us consider an invariant  $I$  of equisingularity. For every nearly convenient Newton diagram  $\Delta$  we put  $I(\Delta) = I(\Delta(f))$  where  $f$  is a nondegenerate reduced power series. According to the theorem quoted above  $I(\Delta)$  is defined correctly (does not depend on  $f$ ). There is a natural problem: calculate  $I(\Delta)$  effectively in terms of  $\Delta$ . The most known result of this kind is due to Kouchnirenko [Kou1976].

To formulate it let us consider for every nearly convenient Newton diagram  $\Delta$  a convex subset  $\tilde{\Delta}$  of  $\mathbf{R}_+^2$  defined to be the intersection of all half-planes containing  $\Delta$  whose boundary is the line extending a face  $S \in \mathcal{N}(\Delta)$

with  $\mathbf{R}_+^2$ . If  $\mathcal{N}(\Delta) = \emptyset$  then by convention  $\tilde{\Delta} = \mathbf{R}_+^2$ . Let  $(a, 0)$  (resp.  $(0, b)$ ) be the point of  $\tilde{\Delta} \cap \{\beta = 0\}$  (resp.  $\tilde{\Delta} \cap \{\alpha = 0\}$ ) closest to the origin. Let us put  $\mu(\Delta) = 2V - a - b + 1$  where  $V$  is the area of  $\mathbf{R}_+^2 \setminus \tilde{\Delta}$ . Then we have

**THEOREM 1.4.** (see [Kou1976], [GarLenPł2007]) *For any power series  $f \in \mathbf{C}\{X, Y\}$   $\mu_0(f) \geq \mu(\Delta(f))$ . The equality holds if and only if  $f$  is nondegenerate.*

Note that Kouchnirenko proved a much more general result concerning isolated singularities in  $n$  dimensions. In the case  $n = 2$  the result is more precise: the equality  $\mu_0(f) = \mu(\Delta(f))$  holds if and only if  $f$  is nondegenerate and we do not need the assumption “ $f$  is convenient”.

Theorem 1.3 can be easily deduced from the famous  $\mu$ -constant theorem [LêR1976] and Kouchnirenko’s result. One can give also a direct, elementary proof [Len2008]. Let us end this section with

**EXAMPLE 1.5.** Let  $f(X, Y) = \sum c_{\alpha\beta} X^\alpha Y^\beta$  ( $\frac{\alpha}{w_1} + \frac{\beta}{w_2} = 1$  where  $w_1, w_2 \geq 2$  are rational numbers) be a weighted homogeneous polynomial of order  $> 1$ . Then  $\mathbf{R}_+^2 \setminus \tilde{\Delta}(f)$  is the triangle with sides  $\alpha = 0, \beta = 0$  and  $\alpha/w_1 + \beta/w_2 = 1$ . If  $f$  is nondegenerate then by Theorem 1.4  $\mu_0(f) = \mu(\Delta(f)) = (w_1 - 1)(w_2 - 1)$  (the Milnor-Orlik formula).

## 2. The jacobian Newton polygon

The following lemma is well-known (see, for example [Del1991] or [Pł2004]).

**LEMMA 2.1.** *Let  $f, g \in \mathbf{C}\{X, Y\}$  be two power series without constant term. Let  $J(f, g) = (\partial f/\partial X)(\partial g/\partial Y) - (\partial f/\partial Y)(\partial g/\partial X)$  be the Jacobian of the pair  $(f, g)$ . Then*

$$(f, J(f, g))_0 = \mu_0(f) + (f, g)_0 - 1.$$

*The right side of the above equality is finite if and only if the left is too.*

Assume that  $l = 0$  is a regular curve. Let  $f = 0$  be a reduced curve such that  $J(f, l)(0, 0) = 0$ . If  $l = 0$  is not a branch of  $f = 0$  then we call  $J(f, l) = 0$  the *polar curve of  $f = 0$  relative to  $l = 0$* . It depends on the power series  $f$  and  $l$ .

If  $l = bX - aY$  is a nonzero linear form then

$$J(f, l) = a(\partial f/\partial X) + b(\partial f/\partial Y)$$

and we speak about the *polar curve relative to the direction  $(a : b) \in \mathbf{P}^1(\mathbf{C})$* . Using Lemma 2.1 it is easy to check the following two properties. We assume  $J(f, l)(0, 0) = 0$ .

**PROPERTY 2.2.** *The regular curve  $l = 0$  is not a branch of the curve  $f = 0$  if and only if  $l = 0$  is not a branch of the polar curve  $J(f, l) = 0$ .*

Recall that two curves are *transverse* if they have no common tangent.

**PROPERTY 2.3.** *If the curves  $l = 0$  and  $f = 0$  are transverse then the curves  $l = 0$  and  $J(f, l) = 0$  are transverse, too.*

In the sequel we assume that  $f = 0$  is a reduced curve and that the regular curve  $l = 0$  is not a branch of  $f = 0$ .

Recall that  $J(f, l)(0, 0) = 0$  and let  $J(f, l) = h_1 \cdots h_s$  be the decomposition of  $J(f, l)$  into irreducible factors. Then the rational numbers

$$\frac{(f, h_j)_0}{(l, h_j)_0}, \quad j = 1, \dots, s$$

are called the *polar invariants* of  $f = 0$  relative to  $l = 0$ . Let  $Q(f, l)$  be the set of polar invariants. If  $J(f, l)(0) \neq 0$  then we put  $Q(f, l) = \emptyset$ . For every polar invariant  $q \in Q(f, l)$  we put

$$A_q = \{j \in [1, s] : \frac{(f, h_j)_0}{(l, h_j)_0} = q\}$$

and

$$J_q = \prod_{j \in A_q} h_j.$$

Thus

$$J(f, l) = \prod_q J_q \quad \text{and} \quad \frac{(f, J_q)_0}{(l, J_q)_0} = q \quad \text{for } q \in Q(f, l).$$

We call  $m_q = (l, J_q)_0$  the *multiplicity* of the polar invariant  $q$ . Using Lemma 2.1 we check

**PROPERTY 2.4.**  $\sum_q m_q = (f, l)_0 - 1, \quad \sum_q m_q q = \mu_0(f) + (f, l)_0 - 1$   
*where the summation is over all  $q \in Q(f, l)$ .*

Let  $\eta_0(f, l) = \sup Q(f, l)$  be the maximal polar invariant ( $\eta_0(f, l) = -\infty$  if  $J(f, l)(0, 0) \neq 0$ ). Property 2.4 implies

**PROPERTY 2.5.** *Suppose that  $(f, l)_0 > 1$ . Then*

$$\frac{\mu_0(f)}{(f, l)_0 - 1} + 1 \leq \eta_0(f, l) \leq \mu_0(f) + 1.$$

Moreover

$$\eta_0(f, l) = \frac{\mu_0(f)}{(f, l)_0 - 1} + 1$$

*if and only if there exists exactly one polar invariant of  $f = 0$  relative to  $l = 0$ .*

From the above property it follows that a regular plane curve  $f = 0$  has exactly one polar invariant, equal to 1 relative to any nontransverse regular curve  $l = 0$ . In the sequel we assume that  $f = 0$  is a singular reduced curve.

Following Teissier [Te1980] we define the *jacobian Newton polygon* by putting

$$\mathcal{Q}(f, l) = \sum_{j=1}^s \left\{ \frac{(f, h_j)_0}{(l, h_j)_0} \right\}.$$

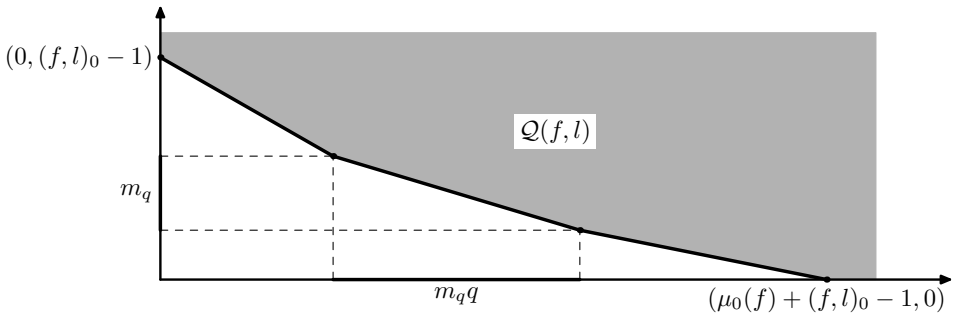
It is easy to see that

$$\mathcal{Q}(f, l) = \sum_q \left\{ \frac{m_q q}{m_q} \right\}.$$

**PROPERTY 2.6.** *The jacobian Newton polygon intersects the axes at points  $(0, (f, l)_0 - 1)$  and  $(\mu_0(f) + (f, l)_0 - 1, 0)$ . All faces of  $\mathcal{Q}(f, l)$  have inclinations strictly greater than 1.*

The above property follows from Property 2.4 and from the following formula

$$(f, h_j)_0 = \inf \left\{ \left( \frac{\partial f}{\partial X}, h_j \right)_0, \left( \frac{\partial f}{\partial Y}, h_j \right)_0 \right\} + (l, h_j)_0 \text{ for } j = 1, \dots, s.$$



**REMARK 2.7.** If  $f = 0$  and  $l = 0$  are transverse then the polar invariants are of the form  $(f, h_j)_0 / \text{ord } h_j$ ,  $j = 1, \dots, s$  (see Property 2.3). The jacobian Newton polygon joins the points  $(0, \text{ord } f - 1)$  and  $(\mu_0(f) + \text{ord } f - 1, 0)$ . Its faces have inclinations greater than or equal to  $\text{ord } f$ . One checks that  $\text{ord } f$  is the polar invariant if and only if the number of tangents  $t(f)$  is strictly greater than 1. Then  $t(f) - 1$  is the multiplicity of  $\text{ord } f$  (see [LenMaPł2003]).

A local isomorphism  $\Phi$  is a pair of power series without constant term such that  $\text{Jac } \Phi(0, 0) \neq 0$ . The jacobian Newton polygon  $\mathcal{Q}(f, l)$  is an analytic invariant of the pair  $(f, l)$ :

**PROPERTY 2.8.** *Let  $\Phi$  be a local isomorphism. Then  $\mathcal{Q}(f \circ \Phi, l \circ \Phi) = \mathcal{Q}(f, l)$ .*



Let  $f = 0$  and  $f' = 0$  be reduced singular curves and let  $l = 0$  and  $l' = 0$  be regular branches such that  $l = 0$  (resp.  $l' = 0$ ) is not a component of  $f = 0$  (resp.  $f' = 0$ ). We will say that the pairs  $f = 0, l = 0$  and  $f' = 0, l' = 0$  are equisingular if there is an equisingularity bijection of the set of branches  $f_i = 0$  of  $f = 0$  and  $f'_i = 0$  of  $f' = 0$  such that  $(f_i, l)_0 = (f'_i, l')_0$  for all  $i = 1, \dots, r$ . The following result is a refinement of Teissier's theorem on invariance of the jacobian Newton polygon [Te1977] in the case of plane curve singularities.

**THEOREM 2.9.** *Suppose that the pairs  $f = 0, l = 0$  and  $f' = 0, l' = 0$  are equisingular. Then*

$$Q(f, l) = Q(f', l').$$

The proof of the above theorem may be given by purely intersection theoretical methods (see [GwLenP1]) based on the Kuo and Lu approach ([KuoLu1977] and Section 3 of this paper).

Now let us note

**COROLLARY 2.10.** *If  $f = 0$  is a reduced singular curve and  $l = 0, l' = 0$  are nonsingular branches transverse to  $f = 0$  then  $Q(f, l) = Q(f, l')$ .*

We write  $Q(f) = Q(f, l)$  provided that  $f = 0$  and  $l = 0$  are transverse and call  $Q(f)$  the jacobian Newton polygon of  $f = 0$ .

**COROLLARY 2.11.** *Suppose that the reduced singular curves  $f = 0$  and  $f' = 0$  are equisingular. Then  $Q(f) = Q(f')$ .*

From the last corollary it follows that the Milnor number  $\mu_0(f)$  and the maximal polar invariant  $\eta_0(f) = \max Q(f)$  are invariants.

**EXAMPLE 2.12.** Let  $f = (Y^3 - X^5)^2 - 9X^{11}$  and  $l = X$ . Then  $(f, l)_0 = \text{ord } f = 6$  i.e.  $f = 0$  and  $l = 0$  are transverse. We get  $J(f, l) = (\partial f / \partial Y) = 6(Y^3 - X^5)Y^2$  and

$$Q(f) = Q(f, l) = \left\{ \frac{(f, Y)_0}{1} \right\} + \left\{ \frac{(f, Y)_0}{1} \right\} + \left\{ \frac{(f, Y^3 - X^5)_0}{3} \right\} = \left\{ \frac{20}{2} \right\} + \left\{ \frac{33}{3} \right\}.$$

The computations of the jacobian Newton polygons in the next two examples were done using Theorem 6.1.

**EXAMPLE 2.13.** ([Len2008]) Let  $f = Y^9 + X^2Y^3 + X^9$  and  $g = Y^5 + XY^4 + X^9$ . Then

$$Q(f) = Q(g) = \left\{ \frac{5}{1} \right\} + \left\{ \frac{27}{3} \right\}$$

but the curves  $f = 0$  and  $g = 0$  are not equisingular. The curve  $f = 0$  has 3 branches while  $g = 0$  has 5.

**EXAMPLE 2.14.** Let  $f = X^3Y^3 + X^2Y^4 + X^8 + Y^7$  and  $g = X^4Y^2 + X^8 + Y^7$ . Then

$$\mathcal{Q}(f) = \left\{ \frac{6 \cdot 2}{2} \right\} + \left\{ \frac{7 \cdot 1}{1} \right\} + \left\{ \frac{8 \cdot 2}{2} \right\}$$

and

$$\mathcal{Q}(g) = \left\{ \frac{6 \cdot 1}{1} \right\} + \left\{ \frac{7 \cdot 3}{3} \right\} + \left\{ \frac{8 \cdot 1}{1} \right\}.$$

We get  $\text{ord } f = \text{ord } g = 6$  and  $\mu_0(f) = \mu_0(g) = 30$ . The Newton polygons  $\mathcal{Q}(f)$  and  $\mathcal{Q}(g)$  have the same inclinations 6, 7, 8 and join the same points (0, 5) and (35, 0) but  $\mathcal{Q}(f) \neq \mathcal{Q}(g)$ .

The following simple proposition gives an effective way of computing the jacobian Newton polygon of the pair  $f(X, Y) = Y^n + a_1(X)Y^{n-1} + \dots + a_n(X)$  (a distinguished polynomial of degree  $n > 1$ ) and  $l(X, Y) = X$  by performing the rational operations on the coefficients  $a_1(X), \dots, a_n(X)$ . It illustrates the leading principle of Teissier’s lectures [Te1976].

**PROPOSITION 2.15.** *Suppose that  $f(X, Y)$  is an  $Y$ -distinguished polynomial of degree  $n > 1$  without multiple factors. Let  $T$  be a new variable and consider the discriminant  $D(X, T) = \text{disc}_Y(f(X, Y) - T)$ . Then  $\mathcal{Q}(f, X) = \Delta_{X,T}(D)$  (the Newton polygon of the discriminant  $D(X, T)$  in coordinates  $X, T$ ).*

**Proof.** Let  $\beta_1(X), \dots, \beta_{n-1}(X) \in \mathbf{C}\{X\}^*$  be the Puiseux roots of equation  $(\partial f / \partial Y)(X, Y) = 0$ . It is easy to see that  $\text{ord } f(X, \beta_1(X)), \dots, \text{ord } f(X, \beta_{n-1}(X))$  is the sequence of polar invariants of  $f = 0$  relative to  $X$  appearing with their multiplicities (if  $h(X, Y) = 0$  is the minimal analytic equation of the series  $\beta(X) \in \mathbf{C}\{X\}^*$  then  $\text{ord } f(X, \beta(X)) = (f, h)_0 / (X, h)_0$ ). On the other hand

$$\begin{aligned} D(X, T) &= \text{disc}_Y(f(X, Y) - T) = \text{resultant}_Y(f(X, Y) - T, \frac{\partial f}{\partial Y}(X, Y)) \\ &= \pm \prod_{j=1}^{n-1} (T - f(X, \beta_j(X))). \end{aligned}$$

We apply the Newton-Puiseux Theorem (see Preliminaries) to  $D(X, T) \in \mathbf{C}\{X, T\}$ . ■

**EXAMPLE 2.16.** Let  $f(X, Y) = (Y^2 - X^3)^2 - X^5Y$ . Then  $f = 0$  and  $X = 0$  are transverse. We have  $D(X, T) = -256T^3 + 256X^6T^2 + 288X^{13}T - 27X^{20} - 256X^{19}$  and

$$\mathcal{Q}(f) = \mathcal{Q}(f, X) = \Delta_{X,T}(D) = \left\{ \frac{6}{1} \right\} + \left\{ \frac{13}{2} \right\}.$$

### 3. Polar invariants and Puiseux series

The following lemma due to Kuo and Lu (see [KuoLu1977], Lemma 3.3) is crucial for the approach to the polar invariants based on Puiseux series (see [Egg1982], [GwP12002], [Wall2003]).

**LEMMA 3.1.** (the Kuo and Lu lemma)

Let  $f = f(X, Y) \in \mathbf{C}\{X, Y\}$  be a  $Y$ -regular power series of order  $n > 1$  and let  $\alpha_1 = \alpha_1(X), \dots, \alpha_n = \alpha_n(X)$  be the Puiseux roots of the equation  $f(X, Y) = 0$ . If  $\beta_1 = \beta_1(X), \dots, \beta_{n-1} = \beta_{n-1}(X)$  are the Puiseux roots of the equation  $(\partial f / \partial Y)(X, Y) = 0$  then for each  $k \in \{1, \dots, n\}$  and for each  $r > 0$

$$\#\{i : \text{ord}(\alpha_i - \alpha_k) = r\} = \#\{i : \text{ord}(\beta_i - \alpha_k) = r\}.$$

A short proof of the above lemma is given in [GwP11991] (see also [GwP12002]).

**REMARK 3.2.** In [KuoLu1977] the following property is stated:

(\*) for given  $\alpha_i, \beta_k$  there exists an  $\alpha_j$  such that  $\text{ord}(\beta_k - \alpha_i) = \text{ord}(\beta_k - \alpha_j) = \text{ord}(\alpha_i - \alpha_j)$ .

To show that (\*) does not hold take  $f(X, Y) = Y(Y - X)(Y - X^2)$ . Then  $\alpha_1 = 0, \alpha_2 = X, \alpha_3 = X^2$  and  $\beta_1 = \frac{2}{3}X + \dots, \beta_2 = \frac{1}{2}X^2 + \dots$ . For  $\alpha_2, \beta_2$  does not exist  $\alpha_j$  such that  $\text{ord}(\beta_2 - \alpha_2) = \text{ord}(\beta_2 - \alpha_j) = \text{ord}(\alpha_2 - \alpha_j)$ .

Note also that property (\*) does not hold under the assumption added in [Gar2000] that  $f(X, 0)f(0, Y) \neq 0$ . To get an example it suffices to replace the series  $f(X, Y)$  considered above by the series  $f(X, Y - X)$ .

The set of all Puiseux series  $\mathbf{C}\{X\}^*$  is an ultrametric space with the order of contact  $O(\varphi, \psi) = \text{ord}(\varphi(X) - \psi(X))$ . That is for any  $\varphi, \psi, \chi \in \mathbf{C}\{X\}^*$ :

- (i)  $O(\varphi, \psi) = +\infty$  if and only if  $\varphi = \psi$ ,
- (ii)  $O(\varphi, \psi) = O(\psi, \varphi)$ ,
- (iii)  $O(\varphi, \psi) \geq \inf\{O(\varphi, \chi), O(\psi, \chi)\}$ .

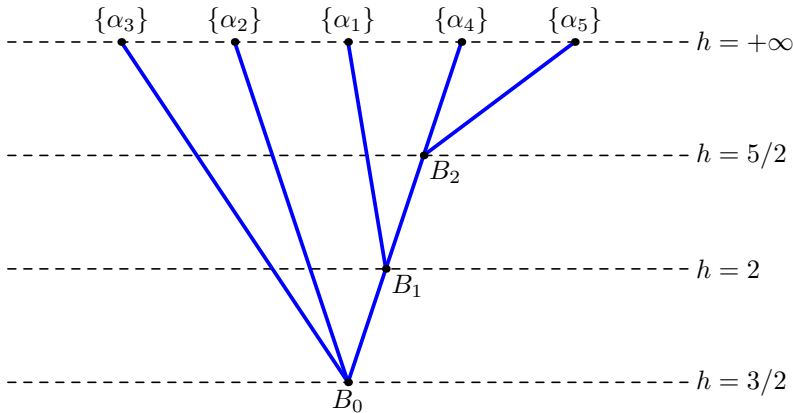
Let  $Z \subset \mathbf{C}\{X\}^*$  be a nonempty finite subset of  $\mathbf{C}\{X\}^*$ . A ball in  $Z$  is a subset  $B \subset Z$  for which there are  $\varphi, \psi \in Z$  such that  $\alpha \in B$  if and only if  $O(\alpha, \varphi) \geq O(\varphi, \psi)$ . We will write  $B = B(\varphi, O(\varphi, \psi))$ . For each ball  $B$  in  $Z$  we define the diameter  $h(B) = \inf\{O(\alpha, \beta) : \alpha, \beta \in B\}$ . Note that if  $B = B(\varphi, O(\varphi, \psi))$  then  $h(B) = O(\varphi, \psi)$ . Let  $\mathcal{B}(Z)$  be the set of balls in  $Z$ . The ordered set  $(\mathcal{B}(Z), \leq)$  where  $B \leq B'$  if and only if  $B \supset B'$  will be called the tree over  $Z$ . If  $B \leq B'$  with  $B \neq B'$  and there is no other ball between  $B$  and  $B'$  then we call  $B'$  a successor of  $B$ . If  $h(B) < +\infty$  i.e. if  $B$  does not reduce to a one-point set then  $B$  has a finite number  $t(B)$  of successors. One has  $t(B) \geq 2$ .

Let  $f = f(X, Y) \in \mathbf{C}\{X, Y\}$  be a  $Y$ -regular power series of order  $n = \text{ord } f(0, Y) \geq 1$ . Assume that  $f$  has no multiple factors and let

$$Z_f = \{\alpha = \alpha(X) \in \mathbf{C}\{X\}^* : \text{ord } \alpha(X) > 0 \text{ and } f(X, \alpha(X)) = 0\}.$$

Thus  $\#Z_f = n$ . The tree over  $Z_f$  will be denoted  $T(f)$  and called the *Kuo-Lu tree model of  $f$*  (see [KuoLu1977] where the balls are called bars and  $h(B)$  is called height of  $B$ ).

**EXAMPLE 3.3.** (see [IzuKoiKuo2002]) Let  $f(X, Y) = (Y - X^2)(Y^2 - X^3)(Y^2 - X^5)$ . Here  $\alpha_1 = X^2$ ,  $\alpha_2 = X^{3/2}$ ,  $\alpha_3 = -X^{3/2}$ ,  $\alpha_4 = X^{5/2}$ ,  $\alpha_5 = -X^{5/2}$  are the roots of  $f(X, Y) = 0$ . Thus  $Z_f = \{\alpha_1, \dots, \alpha_5\}$  and  $O(Z_f \times Z_f) = \{3/2, 2, 5/2, +\infty\}$ . It is easy to check that  $T(f) = \{B_0, B_1, B_2, \{\alpha_1\}, \dots, \{\alpha_5\}\}$  where  $B_0 = Z_f$ ,  $B_1 = \{\alpha_1, \alpha_4, \alpha_5\}$ ,  $B_2 = \{\alpha_4, \alpha_5\}$ . The successors of  $B_0$  are  $B_1$ ,  $\{\alpha_2\}$ ,  $\{\alpha_3\}$ , the successors of  $B_1$  are  $\{\alpha_1\}$  and  $B_2$  and the successors of  $B_2$  are  $\{\alpha_4\}$  and  $\{\alpha_5\}$ . Thus we have  $t(B_0) = 3$ ,  $t(B_1) = 2$ ,  $t(B_2) = 2$ . We can represent the tree  $T(f)$  in the following figure



The balls are represented by points situated on different levels corresponding to the heights  $h \in O(Z_f \times Z_f)$ . We join every ball by continuous lines with its successors.

For each  $\alpha \in Z_f$  and for each ball  $B \in T(f)$  we put  $O(\alpha, B) = \sup\{O(\alpha, \varphi) : \varphi \in B\}$ . Let  $T(f)' = \{B \in T(f) : h(B) < +\infty\}$  and put

$$q(B) = \sum_{\alpha \in Z_f} \inf\{O(\alpha, B), h(B)\}.$$

Note that  $O(\alpha, B) = O(\alpha, \varphi)$  for any  $\varphi \in B$  provided that  $O(\alpha, B) < h(B)$ .

**THEOREM 3.4.** *Let  $f = f(X, Y) \in \mathbf{C}\{X, Y\}$ ,  $n = \text{ord } f(0, Y) > 1$  be a power series without multiple factors. Then*

- (i)  $q \in Q(f, X)$  if and only if  $q = q(B)$  for a ball  $B \in T(f)'$ ,
- (ii)  $m_q = \sum_B (t(B) - 1)$  where the summation is over all  $B \in T(f)'$  such that  $q = q(B)$ .

The above quoted theorem is implicit in [KuoLu1977]. Part (i) was proved in [GwPl2002]. A short proof of (i) and (ii) is given in [GarGw2008].

**EXAMPLE 3.5.** Let us calculate  $Q(f, X)$  for  $f = (Y - X^2)(Y^2 - X^3)(Y^2 - X^5)$ . Using the notation from Example 3.3 we get  $q(B_0) = (\#Z_f)h(B_0) = 5 \cdot (3/2) = 15/2$ ,  $q(B_1) = O(\alpha_2, B_1) + O(\alpha_3, B_1) + (\#B_1)h(B_1) = (3/2) + (3/2) + 3 \cdot 2 = 9$ ,  $q(B_2) = O(\alpha_1, B_2) + O(\alpha_2, B_2) + O(\alpha_3, B_2) + (\#B_2)h(B_2) = 2 + (3/2) + (3/2) + 2 \cdot (5/2) = 10$ . Consequently, we get

$$\begin{aligned} Q(f) = Q(f, X) &= \left\{ \frac{(15/2)(3-1)}{3-1} \right\} + \left\{ \frac{9(2-1)}{2-1} \right\} + \left\{ \frac{10(2-1)}{2-1} \right\} \\ &= \left\{ \frac{15}{2} \right\} + \left\{ \frac{9}{1} \right\} + \left\{ \frac{10}{1} \right\}. \end{aligned}$$

**REMARK 3.6.** In [Len2004] the polar invariants and their multiplicities are computed by using the Newton algorithm.

#### 4. The case of one branch

Let  $f = 0$  be a singular branch. For any regular curve  $l = 0$  the semigroup  $S(f)$  has the  $(f, l)_0$ -minimal system of generators  $\bar{b}_0, \bar{b}_1, \dots, \bar{b}_h$  defined by conditions

- (i)  $\bar{b}_0 = (f, l)_0$ ,
- (ii)  $\bar{b}_k = \min(S(f) \setminus (\mathbf{N}\bar{b}_0 + \dots + \mathbf{N}\bar{b}_{k-1}))$ ,
- (iii)  $S(f) = \mathbf{N}\bar{b}_0 + \dots + \mathbf{N}\bar{b}_h$ .

We will write  $\langle \bar{b}_0, \dots, \bar{b}_h \rangle$  instead of  $\mathbf{N}\bar{b}_0 + \dots + \mathbf{N}\bar{b}_h$ . If  $f = 0$  and  $l = 0$  are transverse then  $(f, l)_0 = \text{ord } f$  and the corresponding system of  $(f, l)_0$ -minimal generators will be denoted by  $\bar{\beta}_0, \bar{\beta}_1, \dots, \bar{\beta}_g$ . Here  $\bar{\beta}_0 = \min(S(f) \setminus \{0\})$ . Let  $n_1, \dots, n_h$  be the integers defined to be

$$n_k = \frac{\text{GCD}(\bar{b}_0, \dots, \bar{b}_{k-1})}{\text{GCD}(\bar{b}_0, \dots, \bar{b}_k)} \quad \text{for } k = 1, \dots, h.$$

Then  $n_k > 1$  for all  $k$ . Now we can state the result due to [Sm1875], [Mer1977] and [Eph1983].

**THEOREM 4.1.** (Smith–Merle–Ephraim) *Suppose that  $f = 0$  is a singular branch and  $l = 0$  a regular curve. Let  $\bar{b}_0, \dots, \bar{b}_h$  be the  $(f, l)_0$ -minimal system of generators of the semigroup  $S(f)$ . Then with the notation introduced above*

$$\mathcal{Q}(f, l) = \sum_{k=1}^h \left\{ \frac{(n_k - 1)\bar{b}_k}{(n_k - 1)n_1 \dots n_{k-1}} \right\}.$$

By convention the empty product which appears for  $k = 1$  is equal to 1.

The sequence of generators can be characterized in purely arithmetical terms. Let us recall (see [Bre1972], [Zariski1973], [Del1994], [GwP1995]).

**THEOREM 4.2.** *Let  $\bar{b}_0, \bar{b}_1, \dots, \bar{b}_h$  be a sequence of strictly positive integers. Then the following two conditions are equivalent.*

- (I) *There is a singular branch  $f = 0$  and a regular curve  $l = 0$  such that  $\bar{b}_0, \bar{b}_1, \dots, \bar{b}_h$  is the  $(f, l)_0$ -minimal system of generators of the semigroup  $S(f)$ ,*
- (II) *the sequence  $\bar{b}_0, \bar{b}_1, \dots, \bar{b}_h$  satisfies the conditions:*
  - (Z<sub>1</sub>) *the sequence  $e_k = \text{GCD}(\bar{b}_0, \dots, \bar{b}_k)$  ( $k = 0, 1, \dots, h$ ) is strictly decreasing and  $e_h = 1$ ,*
  - (Z<sub>2</sub>) *the sequence  $e_{k-1}\bar{b}_k$  ( $k = 1, \dots, h$ ) is strictly increasing.*

**EXAMPLE 4.3.** For any integer  $n \geq 0$  there is a branch  $f = 0$  with the semigroup  $\langle 6, 8, 27 + 6n \rangle$ . By Theorem 4.1 we get

$$\mathcal{Q}(f) = 2 \left\{ \frac{8}{1} \right\} + 3 \left\{ \frac{9 + 2n}{1} \right\}.$$

Using Theorems 4.1 and 4.2 we get

**COROLLARY 4.4.** *Let  $f = 0$  be a singular branch. Then*

- (1)  *$\mathcal{Q}(f, l)$  is a complete invariant of the pair  $f = 0, l = 0$ ;*
- (2)  *$\mathcal{Q}(f)$  is a complete invariant of the branch  $f = 0$ .*

**THEOREM 4.5.** [GarGw2008] *Let  $f = 0$  and  $g = 0$  be two reduced curves such that  $\mathcal{Q}(f) = \mathcal{Q}(g)$ . Suppose that  $f = 0$  is an irreducible curve. Then  $g = 0$  is also irreducible.*

For every sequence  $\bar{b}_0, \dots, \bar{b}_h$  satisfying conditions (Z<sub>1</sub>) and (Z<sub>2</sub>) (in the sequel we call such a sequence (Z)-sequence) we put

$$\mathcal{N}(\bar{b}_0, \dots, \bar{b}_h) = \sum_{k=1}^h \left\{ \frac{(n_k - 1)\bar{b}_k}{(n_k - 1)n_1 \dots n_{k-1}} \right\}.$$

and call  $\mathcal{N}(\bar{b}_0, \dots, \bar{b}_h)$  the Newton diagram associated with the sequence  $\bar{b}_0, \dots, \bar{b}_h$ . Theorems 4.1, 4.5 and Proposition 2.15 give rise to the following

**COROLLARY 4.6.** (Irreducibility Criterion) *Let  $f = Y^n + a_1(X)Y^{n-1} + \dots + a_n(X) \in \mathbf{C}\{X\}[Y]$  be a distinguished polynomial of degree  $n > 1$  without multiple factors. Then  $f$  is irreducible if and only if the Newton diagram of the discriminant  $D(X, T) = \text{disc}_Y(f(X, Y) - T)$  is equal to the Newton diagram  $\mathcal{N}(\bar{b}_0, \dots, \bar{b}_h)$  associated with a  $(Z)$ -sequence  $\bar{b}_0 = n, \bar{b}_1, \dots, \bar{b}_h$ .*

**EXAMPLE 4.7.** (see [Kuo1989] and [Abh1989]). The following two examples are taken from [Kuo1989].

**I.** Let  $f = (X^2 - Y^3)^2 - Y^7$ . Then the curves  $f = 0$  and  $Y = 0$  are transverse and  $\mathcal{Q}(f) = \mathcal{Q}(f, Y) = \left\{ \frac{6}{1} \right\} + \left\{ \frac{14}{2} \right\}$ . To decide if  $f$  is irreducible suppose that  $\mathcal{Q}(f) = \mathcal{N}(\bar{b}_0, \dots, \bar{b}_h)$  for a  $(Z)$ -sequence  $\bar{b}_0, \dots, \bar{b}_h$ . Then  $h = 2$  since  $\mathcal{Q}(f)$  has two faces and  $\bar{b}_0 = \text{ord} f = 4$ . From condition

$$\left\{ \frac{(n_1 - 1)\bar{b}_1}{n_1 - 1} \right\} + \left\{ \frac{(n_2 - 1)\bar{b}_2}{(n_2 - 1)n_1} \right\} = \left\{ \frac{6}{1} \right\} + \left\{ \frac{14}{2} \right\}$$

we get  $\bar{b}_1 = 6$  and  $\bar{b}_2 = 14$ . A contradiction since  $\text{GCD}(\bar{b}_0, \bar{b}_1, \bar{b}_2) = 2$ . Therefore  $f$  is not irreducible.

**II.** Let  $f = (X^2 - Y^3)^2 - Y^5X$ . The curves  $f = 0$  and  $Y = 0$  are transverse and  $\mathcal{Q}(f) = \mathcal{Q}(f, Y) = \left\{ \frac{6}{1} \right\} + \left\{ \frac{13}{2} \right\}$  (see Example 2.16). It is easy to check that  $\mathcal{Q}(f) = \mathcal{N}(4, 6, 13)$  and that  $4, 6, 13$  is a  $(Z)$ -sequence. Therefore  $f$  is irreducible with semigroup  $S(f) = \langle 4, 6, 13 \rangle$ .

### 5. Polar invariants in many branched case

Let  $\varphi, \psi \in \mathbf{C}\{X, Y\}$  be irreducible power series. The *contact coefficient* (in the sense of Hironaka) with respect to a regular curve  $l = 0$  is the rational number

$$h(\varphi, \psi; l) = \frac{(\varphi, \psi)_0}{(l, \psi)_0}.$$

If  $l = 0$  and  $\psi = 0$  are transverse then  $h(\varphi, \psi; l) = (\varphi, \psi)_0 / \text{ord} \psi$  and we write  $h(\varphi, \psi)$  instead of  $h(\varphi, \psi; l)$ .

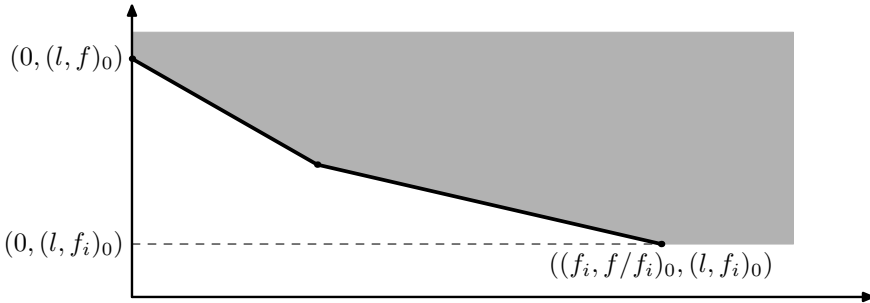
Let  $f = 0$  be a reduced curve with  $r > 1$  branches. To describe the contacts of  $f_i = 0$  with the branches  $f_j = 0, j \neq i$  let us consider the following diagram

$$\mathcal{H}_i(f, l) = \sum_{j=1}^r \left\{ \frac{(f_i, f_j)_0}{(l, f_j)_0} \right\}$$

and the set

$$H_i(f, l) = \left\{ \frac{(f_i, f_j)_0}{(l, f_j)_0} : j \neq i \right\}.$$

Note that the diagram  $\mathcal{H}_i(f, l)$  lies above horizontal axis and has vertices  $(0, (l, f)_0)$  and  $((f_i, f/f_i)_0, (l, f_i)_0)$ . The distance from  $\mathcal{H}_i(f, l)$  to the horizontal axis is equal to  $(l, f_i)_0$ .



We omit the simple proof of the following

**LEMMA 5.1.** *The line with slope  $-1/\tau$  ( $\tau > 0$ ) supporting  $\mathcal{H}_i(f, l)$  intersects the horizontal axis at the point*

$$\left( \sum_{j=1}^r \inf\{(f_i, f_j)_0, \tau(l, f_j)_0\}, 0 \right).$$

Now let

$$q_i(\tau) = \frac{1}{(l, f_i)_0} \sum_{j=1}^r \inf\{(f_i, f_j)_0, \tau(l, f_j)_0\}$$

for  $\tau > 0$  and  $i = 1, \dots, r$ . According to Lemma 5.1 the function  $q_i$  is determined by the diagram  $\mathcal{H}_i(f, l)$  and has an obvious geometric interpretation. The functions  $q_i$  are piecewise linear, continuous and strictly increasing. The following explicit formula for polar quotients of a many-branched curve is due to [GwPl2002].

**THEOREM 5.2.** *Let  $f = f_1 \dots f_r$  be a reduced power series with  $r > 1$  irreducible factors. Then*

$$Q(f, l) = \bigcup q_i(Q(f_i, l) \cup H_i(f, l)).$$

We call the elements of  $q_i(Q(f_i, l) \cup H_i(f, l))$  polar invariants associated with the branch  $f_i = 0$ . A polar invariant can be associated with more than one branch.

The polar invariants associated with the branch  $f_i = 0$  can be interpreted in terms of the Newton diagram  $\mathcal{H}_i(f, l)$  and the jacobian Newton polygon  $Q(f_i, l)$  of the branch  $f_i = 0$ . To this end call a line supporting  $\mathcal{H}_i(f, l)$  distinguished if it extends a face of  $\mathcal{H}_i(f, l)$  or is parallel to a face of  $Q(f_i, l)$ . Then the polar invariants associated to the branch  $f_i = 0$  are exactly the quotients of the form  $\frac{p}{d_i}$  where  $(p, 0)$  is the point of intersection



of a distinguished supporting line with the horizontal axis and  $d_i = (l, f_i)_0$  is the distance from  $\mathcal{H}_i(f, l)$  to this axis.

Let us calculate  $\eta(f, l) = \sup Q(f, l)$ . Using the fact that the functions  $q_i$  are increasing we get

**THEOREM 5.3.** [Pl2001]

$$\eta(f, l) = \max_{i=1}^r \left\{ \max \left\{ \eta(f_i, l), \max_{j \neq i} \frac{(f_i, f_j)_0}{(l, f_j)_0} \right\} + \frac{1}{(l, f_i)_0} \sum_{j \neq i} (f_i, f_j)_0 \right\}.$$

For the applications of the above formula see [GarKP2005].

If  $f = 0$  and  $l = 0$  are transverse then we write  $\mathcal{H}_i(f) = \mathcal{H}_i(f, l)$ .

**EXAMPLE 5.4.** Even when  $f = 0$  and  $g = 0$  are curves with smooth branches the conditions  $\mathcal{H}_i(f) = \mathcal{H}_i(g)$  ( $i = 1, \dots, r$ ) do not imply the equisingularity of  $f = 0$  and  $g = 0$ . Let  $f = f_1 \dots f_{10}$  and  $g = g_1 \dots g_{10}$  where  $f_1 = Y - X - X^2$ ,  $f_2 = Y - X - 2X^2$ ,  $f_3 = Y - X - 3X^2$ ,  $f_4 = Y - 2X - X^2$ ,  $f_5 = Y - 2X - 2X^2$ ,  $f_6 = Y - 2X - 3X^2$ ,  $f_7 = Y - X$ ,  $f_8 = Y - X - X^3$ ,  $f_9 = Y - 2X$ ,  $f_{10} = Y - 2X - X^3$  and  $g_1 = Y - X$ ,  $g_2 = Y - X - X^2$ ,  $g_3 = Y - X - 2X^2$ ,  $g_4 = Y - X - 3X^2$ ,  $g_5 = Y - X - 4X^2$ ,  $g_6 = Y - 2X - 2X^2$ ,  $g_7 = Y - 2X$ ,  $g_8 = Y - 2X - X^3$ ,  $g_9 = Y - 2X - X^2$ ,  $g_{10} = Y - 2X - X^2 - X^3$ . Then  $\mathcal{H}_i(f) = \mathcal{H}_i(g)$  for  $i = 1, \dots, 10$  but  $f = 0$  and  $g = 0$  are not equisingular.

To construct a complete invariant of the pair  $f = 0, l = 0$  the notion of partial polar quotient introduced in [Egg1982] is useful. E. García Barroso characterized the type of equisingularity of the curve by matrices of partial polar quotients (see [Gar2000]).

### 6. Polar invariants and the Newton diagram

We want to calculate the jacobian Newton polygon of a nondegenerate singularity  $f = 0$  in terms of the Newton diagram  $\Delta(f)$ . To formulate the result we need some notions. Let  $f \in \mathbf{C}\{X, Y\}$  be a nonzero power series without constant term. The segment  $S \in \mathcal{N}_f$  is *principal* if  $|S|_1 = |S|_2$ . If a principal segment exists it is unique. Put  $\mathcal{N}'_f = \mathcal{N}_f \setminus \{\text{principal segment}\}$ . For every segment  $S \in \mathcal{N}'_f$  we put  $m(S) = \min(|S|_1, |S|_2) - 1$  if  $1 \leq |S|_1 < |S|_2$  and  $S$  has a vertex on the vertical axis or if  $1 \leq |S|_2 < |S|_1$  and  $S$  has a vertex on the horizontal axis. Moreover we let  $m(S) = \min(|S|_1, |S|_2)$  for all remaining cases.

Let  $\alpha/\alpha(S) + \beta/\beta(S) = 1$  be the equation of the line containing  $S$ . Obviously  $\alpha(S), \beta(S) > 0$  are rational numbers and  $\alpha(S)/\beta(S) = |S|_1/|S|_2$ .

Recall that  $t(f)$  is the number of tangents to  $f = 0$ . If  $f$  is nondegenerate then  $t(f)$  can be read from the Newton diagram  $\Delta(f)$ . We have the following result due to [LenPl2000] (see also [LenMaPl2003]).

**THEOREM 6.1.** *Suppose that  $f$  is a nondegenerate singularity. Then*

$$Q(f) = \left\{ \frac{(\text{ord } f)(t(f) - 1)}{(t(f) - 1)} \right\} + \sum_{S \in \mathcal{N}'_f} \left\{ \frac{\max(\alpha(S), \beta(S)) m(S)}{m(S)} \right\}.$$

We put by convention  $\left\{ \frac{0}{0} \right\} = \mathbf{R}_+^2$  (the zero Newton diagram).

**EXAMPLE 6.2.** Let  $f = \sum c_{\alpha,\beta} X^\alpha Y^\beta$  with summation over all  $(\alpha, \beta) \in \mathbf{N}^2$  such that  $(\alpha/w_1) + (\beta/w_2) = 1$  where  $w_1, w_2 \geq 2$  are rational numbers defines a reduced curve  $f = 0$ . Then  $\eta_0(f) = \max(w_1, w_2)$  by Theorem 6.1. On the other hand  $\mu_0(f) = (w_1 - 1)(w_2 - 1)$  by the Milnor-Orlik formula. Hence the set of weights

$$\{w_1, w_2\} = \left\{ \frac{\mu_0(f)}{\eta_0(f) - 1} + 1, \eta_0(f) \right\}$$

is an invariant of  $f = 0$ .

### 7. Application to pencils of plane curve singularities

When studying the singularities at infinity of polynomials in two complex variables of degree  $N > 1$  one considers the pencils of plane curves of the form  $f_t = f - tl^N$ ,  $t \in \mathbf{C}$  where  $f, l \in \mathbf{C}\{X, Y\}$  are coprime and a regular curve  $l = 0$  is not a component of the local curve  $f = 0$  (see [Eph1983], [GarPł2004], [LenMaPł2003], [Pł2004]). Let  $U \subset \mathbf{C}$  be a Zariski open subset of  $\mathbf{C}$ . We say that the pencil  $(f_t : t \in U)$  is equisingular if the Milnor number  $\mu_0(f_t)$  is constant for  $t \in U$ . This means by  $\mu$ -constant theorem for pencils [Cas2000] that for any  $t_1, t_2 \in U$  the curves  $f_{t_1} = 0$  and  $f_{t_2} = 0$  are equisingular.

**PROPOSITION 7.1.** ([Eph1983], [GarPł2004]) *Let  $f = 0$  be a reduced curve and  $l = 0$  a regular curve which is not a branch of  $f = 0$ . Let  $N > 0$  be an integer. Then*

- (1) *the pencil  $(f - tl^N : t \neq 0)$  is equisingular if and only if  $N \notin Q(f, l)$ ,*
- (2) *the pencil  $(f - tl^N : t \in \mathbf{C})$  is equisingular if and only if  $\eta(f, l) = \sup Q(f, l) < N$ .*

Using the above proposition and a result of Ephraim [Eph1983] we get the following

**PROPOSITION 7.2.** *Let  $f = 0$  be a singular branch,  $l = 0$  a regular one. Let  $(\bar{b}_0, \bar{b}_1, \dots, \bar{b}_h)_0$  be the  $(f, l)_0$ -minimal system of generators of the semigroup  $S(f)$ . Then the following three conditions are equivalent*

- (AM)  $e_{h-1} \bar{b}_h < (\bar{b}_0)^2$ ,
- (I) *all series  $f_t = f - tl^{\bar{b}_0}$ ,  $t \in \mathbf{C}$  are irreducible,*
- (E) *the pencil  $(f_t = f - tl^{\bar{b}_0} : t \in \mathbf{C})$  is equisingular.*

**Proof.** By Theorem 4.1 we have  $\eta_0(f, l) = e_{h-1}\bar{b}_h/\bar{b}_0$ . Therefore (AM) is equivalent to the inequality  $\eta_0(f, l) < \bar{b}_0$  and (AM)  $\Leftrightarrow$  (E) follows from Proposition 7.1(2). Obviously (E)  $\Rightarrow$  (I), the implication (I)  $\Rightarrow$  (E) is due to Ephraim [Eph1983], Corollary 2.2. ■

Note that (AM) is the famous Abhyankar–Moh inequality (see [AbhMoh1975], [GwP1995], [Cas2000]). For more applications of polar invariants to the singularities at infinity we refer the reader to [GarP12004], [GwP12005], [P12002] and to the papers cited in these articles.

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