BULLETIN OF THE POLISH
ACADEMY OF SCIENCES
MATHEMATICS
Vol. 46, No. 1, 1998

ALGEBRAIC GEOMETRY

Effective Nullstellensatz on Analytic and Algebraic Varieties

by

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Presented by S. Łojasiewicz on June 2, 1997

Summary. Sharp estimates of the Nullstellensatz exponent on analytic and algebraic sets are given.

1. Local case. Let X be an analytic subset of an open subset Ω of the space $\mathbb{C}^m$ of pure dimension k. Denote by Δ the unit disc in $\mathbb{C}$

LEMMA 1.1. *Suppose that $\varphi = (\varphi_1, \ldots, \varphi_k) : X \to \Delta^k$ is a proper holomorphic mapping of multiplicity μ. If $1 \leqslant l \leqslant k$, and a holomorphic function $g : X \to \mathbb{C}$ vanishes on the set $\{x \in X : \varphi_1(x) = \ldots = \varphi_l(x) = 0\}$, then there exist $h_1, \ldots, h_l$ holomorphic on X such that $g^\mu = \varphi_1 h_1 + \ldots + \varphi_l h_l$.*

Proof. Let us consider $Y = \{(\varphi(x), g(x)) \in \Delta^k \times \mathbb{C} : x \in X\}$. It is easy to check that there exists a polynomial $P \in \mathcal{O}(\Delta^k)[T]$ distinguished in T of degree μ such that $Y = P^{-1}(0)$. Let us write

$$P((y,z),T) = T^\mu + a_1(y,z)T^{\mu-1} + \ldots + a_\mu(y,z)$$

where $(y,z) \in \Delta^l \times \Delta^{k-l} = \Delta^k$. Since $Y \cap \{((y,z),t) \in \Delta^k \times \mathbb{C} : y = 0\} = (\{0\}^l \times \Delta^{k-l}) \times \{0\}$, then $a_1, \ldots a_\mu$ belong to the ideal of the set $\{0\}^l \times \Delta^{k-l}$ in the ring $\mathcal{O}(\Delta^k)$. Therefore we obtain for each $j = 1, \ldots, \mu$

$$a_j(y,z) = y_1 a_{j,1}(y,z) + \ldots + y_l a_{j,l}(y,z), \quad (y,z) \in \Delta^k$$

where $a_{j,i} \in \mathcal{O}(\Delta^k)$ for $i = 1, \ldots, l$. Now, observe that

$$P(((\varphi_1(x), \ldots \varphi_l(x)), (\varphi_{l+1}(x), \ldots \varphi_k(x))), g(x)) = 0,$$

1991 MS Classification: 14E99, 32B99.
Key words: local multiplicity, geometric degree, degree of an algebraic set.
(*) Research supported by KBN Grant 2 P03A 079 08.
(**) Research supported by KBN Grant 2 P03A 061 08.

and by a simple calculation we get the required result.

Now, suppose that X is an analytic subset of $\Omega \subset \mathbb{C}^m$ of pure dimension k and that holomorphic functions $f_1, \dots f_l$ on X realize a set-theoretic complete intersection. This means that $\dim f^{-1}(0) = k - l$, where $f = (f_1, \dots, f_l) : X \to \mathbb{C}^l$. Observe that the graph of f meets properly $\Omega \times \{0\}^l$ in the space $\Omega \times \mathbb{C}^l$. Then the intersection product of above sets is an analytic cycle on $\Omega \times \mathbb{C}^l$ defined by the formula $f \cdot (\Omega \times \{0\}^l) = \sum_C \alpha_C C$, where the summation extends over all analytic components $C = Z \times \{0\}^l$ of $f^{-1}(0) \times \{0\}^l$ and $\alpha_C = i(f \cdot (\Omega \times \{0\}^l), C)$ denotes the intersection multiplicity along the component C in the sense of Draper ([5], Def. 4.5, see also [16], [17]). In this situation *the cycle of zeroes of* f (cp. [16]) is defined to be

$$Z_f = \sum_Z \alpha_{Z \times \{0\}^l} Z$$

where the summation extends over all irreducible components of $f^{-1}(0)$.

For an irreducible analytic subset Y of Ω and $a \in \Omega$ we denote by $\nu(Y, a)$ the degree of Y at the point a (cf. [5], p. 194). This degree is equal to the classical algebraic Samuel multiplicity, and so called, Lelong number of Y at the point a. We will consider a natural extension of this definition to the case of an arbitrary analytic cycle. Namely, if $A = \sum_Y \alpha_Y Y$ is an analytic cycle in $\Omega \subset \mathbb{C}^m$, then the sum

$$\nu(A, a) = \sum_Y \alpha_Y \nu(Y, a)$$

is well defined and we call it the *degree* of the cycle A at the point a.

THEOREM 1.2. *Suppose that X is an analytic subset of $\Omega \subset \mathbb{C}^m$ of pure dimension k, holomorphic functions $f_1, \dots f_l$ on X realize a set-theoretic complete intersection and denote $f = (f_1, \dots, f_l) : X \to \mathbb{C}^l$. If holomorphic function $g : X \to \mathbb{C}$ vanishes on the set $f^{-1}(0)$ and $a \in \Omega$, then there exist an open neighbourhood V of a in the set X and $h_1, \dots, h_l \in \mathcal{O}(V)$ such that*

$$[g(x)]^\mu = f_1(x)h_1(x) + \dots + f_l(x)h_l(x) \quad \textit{for } x \in V$$

where $\mu = \nu(Z_f, a)$ is the degree of the cycle of zeroes of f at a.

Proof. Since $\dim f^{-1}(0) = k - l$, then without loss of generality we can assume that $a = 0$ and $\{0\}^{k-l} \times \mathbb{C}^{m-(k-l)}$ properly intersects cycle Z_f at 0 with (minimal possible) multiplicity μ. Let us consider the mapping

$$\varphi : X \ni (x_1, \dots, x_m) \to (f_1(x), \dots, f_l(x), x_1, \dots, x_{k-l}) \in \mathbb{C}^k.$$

Since 0 is an isolated point of $\varphi^{-1}(0)$ then there exists V its open neighbourhood such that $\varphi \mid V : V \to \varphi(V) \subset \mathbb{C}^k$ is proper and $\varphi^{-1}(0) \cap V = \{0\}$. Moreover, we can assume that $\varphi(V) = \Delta^k$. By Lemma 1.1 it is sufficient to show that the mapping φ has multiplicity μ.

Denote $Y = f \mid X \cap V$ and observe that the multiplicity of φ is equal to the multiplicity of the restricted projection $\pi \mid Y : Y \ni (x_1, \dots x_m, y_1, \dots, y_l) \to (y_1, \dots, y_l, x_1, \dots x_{k-l}) \in \Delta^k$. The multiplicity of $\pi \mid Y$ is equal to the multiplicity of the intersection of Y and $\pi^{-1}(0)$. By ([16], Thm 2.2) we have $Y \cdot \pi^{-1}(0) = (Y \cdot (V \times \{0\}^l)) \cdot_{V \times \{0\}^l} (\{0\}^{k-l} \times \mathbb{C}^{m-(k-l)}) = Z_f \cdot (\{0\}^{k-l} \times \mathbb{C}^{m-(k-l)}) = \mu\{0\}$ and the proof is completed.

2. The Max Noether Theorem on algebraic varieties. In this part we present two versions of the Max Noether Theorem. We shall use them in the next parts to extend previous local result to the case of polynomial mappings on algebraic set.

Let X be an algebraic subset of the space $\mathbb{C}^m$ and let $Q, P_1, \dots, P_s$ be polynomials on X. We say that the system $Q, P_1, \dots, P_s$ satisfies Noether's condition at $a \in X$ if the germ of Q at a belongs to the ideal generated by the germs of $P_1, \dots, P_s$ at a in the ring $\mathcal{O}_{X,a}$. Note that we can replace the ring $\mathcal{O}_{X,a}$ in the above definition by the ring $\mathcal{R}_{X,a}$ of germs of rational functions regular at a ([7], Prop. 1, p. 462).

PROPOSITION 2.1. *If for each point $a \in X$ the system $Q, P_1, \dots, P_s$ satisfies Noether's condition at a then there exist polynomials $R_1, \dots, R_s$ on X such that*

$$Q = P_1 R_1 + \dots + P_s R_s.$$

P r o o f. Let us take generators $G_1, \dots, G_q$ of the ideal $I(X)$ in the ring $\mathbb{C}[x_1, \dots, x_m]$. By Serre's Lemma ([7], VII.15.2) at each point $a \in X$ the germs of $G_1, \dots, G_q$ generate the ideal $I(X_a)$ in the ring $\mathcal{O}_{\mathbb{C}^m,a}$. Denote by $\tilde{Q}, \tilde{P}_1, \dots, \tilde{P}_s$ arbitrary extensions of $Q, P_1, \dots P_s$ to the whole space $\mathbb{C}^m$. It is easy to verify that the germs of the system $G_1, \dots, G_q, \tilde{P}_1, \dots, \tilde{P}_s$ generate the germ of $\tilde{Q}$ at each point $a \in \mathbb{C}^m$ in $\mathcal{O}_{\mathbb{C}^m,a}$. By the main theorem of [14] there exist polynomials $H_1, \dots, H_{s+q}$ on $\mathbb{C}^m$ such that

$$\tilde{Q} = \tilde{P}_1 H_1 + \dots + \tilde{P}_s H_s + G_1 H_{s+1} + \dots + G_q H_{q+s}$$

Now $R_1 = H_1 \mid X, \dots, R_s = H_s \mid X$ satisfy the assertion of the Proposition 2.1.

Now suppose that $X = \mathbb{C}^m$ and a polynomial mapping $P = (P_1, \dots, P_l) : \mathbb{C}^m \to \mathbb{C}^s$ realizes a set-theoretic complete intersection in the space $\mathbb{C}^m$ i.e. $\dim P^{-1}(0) = m - s \geqslant 0$. Let $P^{-1}(0) = \bigcup_{j=1}^r Z_j$ be a decomposition of $P^{-1}(0)$ into irreducible components.

PROPOSITION 2.2. *Let us suppose that there exist points $a_j \in Z_j$ such that the system $Q, P_1, \dots, P_s$ satisfies Noether's condition at a_j for $j = 1, \dots, r$.*

Then there exist polynomials $R_1, \dots, R_s$ *such that*

$$Q = P_1R_1 + \dots + P_sR_s \quad \textit{in } \mathbb{C}^m.$$

P r o o f. Let I be the ideal generated by $P_1, \dots, P_s$ in the ring of polynomials in m variables. We may assume that $1 \notin I$. Let $I = I_1 \cap \dots \cap I_r$ be a reduced primary decomposition of I. By Maculay's theorem on complete intersections the ideals $J_1 = \sqrt{I_1}, \dots, J_r = \sqrt{I_r}$ are minimal prime ideals of I, and so $J_j = \text{ideal}(Z_j)$. Since $Q, P_1, \dots, P_s$ satisfies Noether's condition at a_j, there exist polynomials S_j such that $S_j(a_j) \neq 0$ and

$$S_jQ = P_1R_{1j} + \dots + P_sR_{sj}$$

in the ring of polynomials for $j = 1, \dots, r$. Therefore $S_jQ \in I_j, S_j \notin J_j$ and $Q \in I_j$ for $j = 1, \dots, r$. Consequently $Q \in I$.

3. Nullstellensatz on algebraic varieties. Let us start with the following basic result

THEOREM 3.1. *Suppose that* X *is an algebraic subset of* $\mathbb{C}^m$ *of pure dimension* k*, polynomial functions* $F_1, \dots F_l$ *on the set* X*, realize a set-theoretic complete intersection. If a polynomial* $G : X \to \mathbb{C}$ *vanishes on the set* $F^{-1}(0)$*, where* $F = (F_1, \dots, F_l)$*, then there exist polynomials* $H_1, \dots, H_l$ *on* X *such that*

$$G^\mu = F_1H_1 + \dots + F_lH_l$$

where $\mu = \deg F$ *is the degree of the graph of* F.

P r o o f. By Theorem 1.2 and Proposition 2.1 it is sufficient to show that $\nu(Z_F, a) \leqslant \deg F$ for every $a \in F^{-1}(0)$. To prove this, fix an a and an affine subspace Λ of $\mathbb{C}^m$ of dimension $m - (k - l)$ such that $a \in \Lambda$, $\dim(F^{-1}(0) \cap \Lambda) = 0$ and $\deg Z_F \cdot \Lambda = \deg Z_F$. Since $(Z_F \cdot \Lambda) \times \{0\}^l = F \cdot (\Lambda \times \{0\}^l)$, we have $\nu(Z_F, a) \leqslant \deg(F \cdot (\Lambda \times \{0\}^l)) \leqslant \deg F$ and the proof is complete.

Now for every algebraic subset X of $\mathbb{C}^m$ we define the extended degree of X by the formula

$$\delta(X) = \sum_{i=1}^{r} \deg X_i$$

where

$$X = \bigcup_{j=1}^{r} X_j$$

is the decomposition of X into irreducible components. Thus $\delta(X) = \deg X$ when X is an algebraic set of pure dimension. Basic properties of $\delta(X)$ are given in [7], pp. 419–420.

THEOREM 3.2 (cf. [3], Proposition 2). *Let X be an algebraic subset of $\mathbb{C}^m$ and let $P, Q : X \to \mathbb{C}$ be polynomial functions such that $P^{-1}(0) \subset Q^{-1}(0)$. Then there exists a polynomial $A : X \to \mathbb{C}$ such that*

$$Q^{\mu} = AP$$

where $\mu = \delta(P)$ is the extended degree of the graph of P.

Proof. Let $X = \bigcup_{i=j}^{r} X_j$ be the decomposition of X into irreducible components. Fix j and consider $P \mid X_j : X_j \to \mathbb{C}$. If $P \mid X_j \neq 0$ then by Thm 3.1 there exists $A_j : X \to \mathbb{C}$ such that $Q^{\mu_i} - A_j P \mid X_j = 0$ for $\mu_j = \deg(P \mid X_j)$. If $P \mid X_j = 0$ then we have $Q \mid X_j = 0$ and so $Q^{\mu_j} - A_j P \mid X_j = 0$ for $A_j = 1$. Therefore

$$\prod_{j=1}^{r}(Q^{\mu_j} - A_j P) = 0 \quad \text{on } X.$$

Consequently $Q^{\mu} = AP$ on X with $\mu = \sum_{j=1}^{r} \mu_j = \delta(P)$.

For a system of polynomials $P_1, \ldots, P_s$ on $\mathbb{C}^m$ denote by $(P_1, \ldots, P_s)$ the ideal generated by $P_1, \ldots, P_s$ in the ring of polynomials. Let $N(P_1, \ldots, P_s)$ be the smallest integer $N \geqslant 1$ such that $Q^N \in (P_1, \ldots, P_s)$ for every $Q \in \sqrt{(P_1, \ldots, P_s)}$. We put $N(\emptyset) = 1$.

PROPOSITION 3.3. *Let $1 \leqslant s \leqslant m+1$ and $d_i = \deg P_i > 0$ for $i = 1, \ldots, s$. Then:*

$$N(P_1, \ldots, P_s) \leqslant (d_1 \cdot \ldots \cdot d_s) N(P_1, \ldots, P_{s-1})$$

for $1 \leqslant s \leqslant m$, and

$$N(P_1, \ldots, P_{m+1}) \leqslant (d_1 \cdot \ldots \cdot d_m) N(P_1, \ldots, P_m)$$

provided $d_{m+1} = \min_{i=1}^{m+1}(d_i)$.

Proof. If $s = 1$ then Proposition 3.3 follows immediately from Theorem 3.2. Let $s > 1$ and $X = \bigcap_{i=1}^{s-1} P_i^{-1}(0)$. We may assume that $X \neq \emptyset$. By properties of extended degree ([7], p. 420) we get $\delta(X) \leqslant d_1 \cdot \ldots \cdot d_{s-1}$. If $Q \in \sqrt{(P_1, \ldots, P_s)}$ then $X \cap P_s^{-1}(0) \subset X \cap Q^{-1}(0)$ and Theorem 3.2 implies $Q^{\mu} = AP_s$ with $\mu = \delta(P_s \mid X) \leqslant \delta(X) d_s \leqslant d_1 \cdot \ldots \cdot d_s$. Therefore $Q^{\mu} - AP_s \in \sqrt{(P_1, \ldots, P_{s-1})}$, $(Q^{\mu} - AP_s)^{N(P_1, \ldots, P_{s-1})} \in (P_1, \ldots, P_{s-1})$ and the first part of (3.3) follows.

In the case $s = m+1$ we must change a little our arguments. After having replaced $P_1, \ldots, P_m$ by $P_1 - c_1 P_{m+1}, \ldots, P_m - c_m P_{m+1}$ with generic constants $c_1, \dots, c_m$ we may assume that P_{m+1} is integral over the ring $\mathbb{C}[P_1, \dots, P_m]$. Consequently the set $P_{m+1}(X)$ is finite and we check easily that $\mu = \delta(P_{m+1} \mid X) \leqslant \delta(X) \leqslant d_1 \cdot \ldots \cdot d_m$. Then we use Theorem 3.2 like in the first part of the proof, and proposition follows.

Let $d = \max_{i=1}^{s} d_i$. By induction we get from Proposition 3.3

$$(*) \qquad N(P_1, \dots, P_s) \leqslant d^{\frac{s(s+1)}{2}} \quad \text{if } 1 \leqslant s \leqslant m.$$

If $s = m+1$ then we get by $(*)$ and the second part of Proposition 3.3

$$(**) \qquad N(P_1, \dots, P_{m+1}) \leqslant d^m N(P_1, \dots, P_m) \leqslant d^m d^{\frac{m(m+1)}{2}} = d^{\frac{m(m+3)}{2}}$$

If $s > m+1$ then $\sqrt{(P_1, \dots, P_s)} = \sqrt{(R_1, \dots, R_{m+1})}$ where R_i are linear combinations of $P_1, \dots, P_s$ (cf. [9]) and $(*)$, $(**)$ imply

$$(***) \qquad N(P_1, \dots, P_s) \leqslant d^{\frac{m(m+3)}{2}} \quad \text{for every } s \geqslant 1.$$

This kind of estimate was obtained in the main theorem of [3]. Let us recall that from the fundamental Kollár result it follows $N(P_1, \dots, P_s) \leqslant d^m$ if $d_i \neq 2$, $i = 1, \dots, s$.

4. Complete intersection. Let $P = (P_1, \dots, P_l) : \mathbb{C}^m \to \mathbb{C}^l$ realize a set-theoretic complete intersection in the space $\mathbb{C}^m$ and let $Z_P = \sum_{j=1}^{r} \mu_j X_j$ be the cycle of zeroes of the polynomial mapping P. Put $d_i = \deg P_i$ for $i = 1, \dots, l$

PROPOSITION 4.1. *With the above assumption*:

(1) $\sum_{j=1}^{r} \mu_j \deg X_j \leqslant d_1 \cdot \ldots \cdot d_l$,

(2) *if polynomials* $R_1, \dots, R_r$ *satisfy the conditions* $R_j \mid X_j = 0$ *for* $j = 1, \dots r$, *then* $R_1^{\mu_1} \cdot \ldots \cdot R_r^{\mu_r} \in (P_1, \dots, P_l)$.

Proof. Let $L = (L_1, \dots, L_{m-l})$ be an affine mapping such that:

(a) $\#(X_j \cap L^{-1}(0)) = \deg X_j$ for $j = 1, \dots, s$

(b) if $a \in X_j \cap L^{-1}(0)$ then a is a regular point of X and $\operatorname{mult}_a(P, L) = \mu_j$ for $j = 1, \dots, s$

We have

$$\sum_{a \in (P,L)^{-1}(0)} \operatorname{mult}_a(P, L) = \sum_{j=1}^{s} \sum_{a \in X_j \cap L^{-1}(0)} \operatorname{mult}_a(P, L) = \sum_{j=1}^{s} \mu_j \deg X_j.$$

On the other hand $\sum_{a \in (P,L)^{-1}(0)} \operatorname{mult}_a(P, L) \leqslant d_1 \cdot \ldots \cdot d_l$ by Bezout's Theorem (cf. [7]). Therefore property (1) follows.

To prove (2) it suffices by (2.2) to verify that

$$R_1^{\mu_1} \cdot \ldots \cdot R_r^{\mu_r} \in (P_1, \dots, P_l)\mathcal{O}_{\mathbb{C}^m, a}$$

for $a \in X_j \cap L^{-1}(0)$ and $j = 1, \dots, s$.

Let $a \in X_j \cap L^{-1}(0)$. Then a is a regular point of X and there is a neighbourhood $\Omega_a \subset \mathbb{C}^m$ of a such that $X \cap \Omega_a = X_j \cap \Omega_a$. Consequently $R_j \mid X \cap \Omega_a = 0$ and we get $R_j^{\mu_j} \in (P_1, \ldots, P_l)\mathcal{O}_{\mathbb{C}^m, a}$ by Lemma 1.1 applied to the mapping $(P, L) : \mathbb{C}^m \to \mathbb{C}^m$ at the point a.

THEOREM 4.2. *Let us suppose that* $P = (P_1, \ldots, P_l) : \mathbb{C}^m \to \mathbb{C}^l$ *realize a set-theoretic complete intersection in the space* $\mathbb{C}^m$. *Then for every polynomial* $Q : \mathbb{C}^m \to \mathbb{C}$ *such that* $\deg Q > 0$ *we have*

$$N(P_1, \ldots, P_l, Q) \leqslant (\deg Z_P)(\deg Q).$$

Proof. Let $Z_P = \sum_{j=1}^{r} \mu_j X_j$ and $R \in \sqrt{(P_1, \ldots, P_l, Q)}$. Then $Q^{-1}(0) \cap X_j \subset R^{-1}(0) \cap X_j$ for $j = 1, \ldots r$. By Theorem 3.2 there exist polynomials $A_j : \mathbb{C}^m \to \mathbb{C}$ such that

$$R^{(\deg X_j)(\deg Q)} = A_j Q \quad \text{on } X_j \quad \text{for } j = 1, \ldots, r.$$

By Proposition 4.1 we have

$$\prod_{j=1}^{r} (R^{(\deg X_j)(\deg Q)} - A_j Q)^{\mu_j} \in (P_1, \ldots, P_l).$$

Hence

$$R^{(\deg Z_P)(\deg Q)} \in (P_1, \ldots, P_l),$$

and the proof is complete.

From Proposition 4.1 and Theorem 4.2 we get

COROLLARY 4.3 (cf. [6], Thm 1.5). *Let* $P_1, \ldots, P_{l+1}$ *be a sequence of polynomials on* $\mathbb{C}^m$ *such that* $\dim \bigcap_{i=1}^{l} P_i^{-1}(0) = m - l \geqslant 0$ *and let* $d_i = \deg P_i > 0$ *for* $i = 1, \ldots, s+1$. *Then* $N(P_1, \ldots, P_{s+1}) \leqslant d_1 \cdot \ldots \cdot d_{s+1}$.

Let us note that if P_1, P_2, P_3 are polynomials of degrees $d_1, d_2, d_3 > 0$ then:

(a) $N(P_1, P_2) \leqslant d_1 d_2$,

(b) $N(P_1, P_2, P_3) \leqslant d_1 d_2 d_3$.

Indeed (a) follows immediately from Corollary 4.3.

To check (b) observe that we may assume that polynomials are irreducible (one checks easily that $N(P_1' P_1'', P_2, P_3) \leqslant N(P_1', P_2, P_3) + N(P_1'', P_2, P_3)$). If $1 \in (P_1, P_2)$ then we have $N(P_1, P_2, P_3) = 1$. In the case $P_1^{-1}(0) = P_2^{-1}(0)$ (b) is a simple consequence of (a). Finally, in the only non-trivial case is when the set $P_1^{-1}(0) \cap P_2^{-1}(0)$ is non-empty of dimension $m - 2$, (b) follows from Corollary 4.3.

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REFERENCES

[1] R. Achilles, P. Tworzewski, T. Winiarski, *On improper isolated intersection in complex analytic geometry*, Ann. Polon. Math., **51** (1990) 21–36.

[2] W. D. Brownawell, *Bound for the degrees in the Nullstellensatz*, Annals of Math., **126** (1987) 577–591.

[3] L. Caniglia, A. Galligo, J. Heinz, *Borne simple exponentielle pour le degrés dans le théorème des zéros sur un corps de charactéristique quelconque*, C. R. Acad. Sci. Paris, **307**, Serié 1, (1988) 255–258.

[4] P. Cassou-Noguès, A. Płoski, *Un théorème des zéros effectif*, Bull. Pol. Ac.: Math., **44** (1996) 61–70.

[5] R. Draper, *Intersection theory in analytic geometry*, Math. Ann., **180** (1969) 175–204.

[6] J. Kollàr, *Sharp effective Nullstellensatz*, J. Amer. Math. Soc., **1** (1988) 963–975.

[7] S. Łojasiewicz, *Introduction to Complex Analytic Geometry*, Birkhäuser Verlag, Basel 1991.

[8] D. Mumford, *Algebraic Geometry I, Complex Projective Varieties*, Springer Verlag, 1976.

[9] O. Perron, *Beweis and Verschärfung eines Satzes von Kronecker*, Math. Ann., **118** (1943) 441–448.

[10] A. Płoski, *On the growth of proper polynomial mappings*, Ann. Polon. Math., **45** (1985) 297–309.

[11] A. Płoski, *An Inequality for Polynomial Mappings*, Bull. Pol. Ac.: Math., **40** (1992) 265–269.

[12] B. Shiffman, *Degree Bounds for the Division Problem in Polynomial Ideals*, Michigan Math. J., **36** (1989) 163–171.

[13] B. Teissier, *Résultats récents d'algébre commutative effective*, Séminaire Bourbaki 1989–90 No. 718, Astérisque 189–190, 107–131.

[14] P. Tworzewski, *A note on the Max Noether Theorem*, Bull. de la Soc. des Sciences et des Letters de Łódź, **39**, **7**, No. 61 (1989) 1–3.

[15] P. Tworzewski, *Intersection theory in complex analytic geometry*, Ann. Polon. Math., **62** (1995) 177–191.

[16] P. Tworzewski, T. Winiarski, *Cycles of Zeroes of Holomorphic Mappings*, Bull. Pol. Ac.: Math., **37** (1989) 95–101.

[17] T. Winiarski, *Continuity of total number of intersection*, Ann. Polon. Math., **47** (1986) 155–178.