

Bezout's theorem for affine curves with one branch at infinity

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Abstract. Let us assume that each of the two affine curves $C, D \subset \mathbb{C}^2$ has one branch at infinity and that C and D intersect in a finite number k of points counted with multiplicities. Put $m = \deg C$, $n = \deg D$ and $(m, n) = g. c. d. (m, n)$.

Then $k \equiv 0 \left(\text{mod } \frac{m}{(m, n)} \text{ or } \frac{n}{(m, n)} \right)$.

1. In the paper [5] van der Kulk gave a simple proof of Jung's Automorphism Theorem (cf. [4]). Following his ideas we prove in this note a result which may be considered a version of Bezout's theorem. Let us recall some definitions. A polynomial automorphism of \mathbb{C}^2 or, briefly, an automorphism is a bijective polynomial mapping of \mathbb{C}^2 onto \mathbb{C}^2 with a polynomial inverse. The restriction of an automorphism to a line induces an embedding i.e. an injective mapping from \mathbb{C} to \mathbb{C}^2 with nowhere vanishing derivative. It will be convenient to regard \mathbb{C}^2 as the projective plane P^2 without the line at infinity L . For any affine curve $C \subset \mathbb{C}^2$ we denote by \bar{C} the closure of C in P^2 . In the study of automorphisms the following notion arises naturally (cf. [2]):

Definition. The affine curve C is said to have one branch at infinity if its projective closure \bar{C} has only one point which is not in \mathbb{C}^2 and the analytic germ of \bar{C} at this point is irreducible.

It is easy to check that an affine curve which is the image of C under an embedding has one branch at infinity, consequently the images of lines under an automorphism have also this property.

Our main theorem is

THEOREM 1.1. *Let us assume that each of the two affine curves $C, D \subset \mathbb{C}^2$ has one branch at infinity and that C and D intersect in a finite number k of points counted with multiplicities. Put $m = \deg C$, $n = \deg D$ and $(m, n) = g. c. d. (m, n)$.*

Then

$$k \equiv 0 \left(\text{mod } \frac{m}{(m, n)} \text{ or } \frac{n}{(m, n)} \right)$$

In the proof (1.1) we use van der Kulk's method: first we prove a purely local property of the intersection multiplicity of germs of plane curves (cf. Proposition 3.1) below and then we get our result via classical Bezout's theorem. We shall present the proof of (1.1) at the end of this note, now we state the

COROLLARY 1.2 (cf. [5]). *Let the mapping $z \rightarrow (P(z), Q(z))$ be a polynomial automorphism. Then of the two integers $m = \deg P$, $n = \deg Q$ one divides the other.*

Proof. The curves $C = P^{-1}(0)$ and $D = Q^{-1}(0)$ of degrees m and n are images of C under the inverse of the automorphism $z \rightarrow (P(z), Q(z))$, moreover they intersect at exactly one point with multiplicity 1. Hence the assumptions of (1.1) are fulfilled with $k = 1$ and we have $\frac{m}{(m, n)} = 1$ or $\frac{n}{(m, n)} = 1$ which proves the corollary.

From the above corollary one can easily get Jung's Automorphism Theorem (for the details see [5]). The Abhyankar Moh Embedding Theorem (cf. [2]) which serves the same purpose is a deeper result and harder to prove.

2. In this section we recall a classical formula for the intersection multiplicity of two irreducible germs of curves. Let Γ be an irreducible germ of a curve at $0 \in \mathbb{C}^2$, we choose the coordinates x, y so that the tangent to Γ at $0 \in \mathbb{C}^2$ has the equation $y = 0$. Then by the Weierstrass preparation theorem Γ has its defining ideal generated by the distinguished polynomial

$$f(x, y) = y^p + \sum_{j=1}^p a_j(x) y^{p-j},$$

where $a_j(x)$ are convergent power series in x whose order is strictly greater than j . The equation $f(x, y) = 0$ has a solution for y as a power series in $x^{1/p}$:

$$y = \sum_{1 \leq i \leq N} a_i x^{\frac{p_i}{p}}, \quad p < p_1 < p_2 < \dots,$$

where $0 \leq N \leq +\infty$ and $a_i \neq 0$ for all $1 \leq i \leq N$. If $N = 0$, then the sum is equal to zero and $f(x, y) = y$. In the sequel we put $p_0 = p$ and call $(p_i; 0 \leq i \leq N)$ the sequence of Puiseux exponents of Γ with respect to the coordinates x, y . If $N < +\infty$, we define additionally $p_{N+1} = +\infty$ with usual conventions on the symbol $+\infty$. Let us recall that $p_0 = \text{ord } \Gamma$ (order of Γ) and that p_1 is equal to the intersection multiplicity of Γ and its tangent $y = 0$. In the theorem below we assume that the sum of an empty family is equal to zero.

THEOREM 2.1 (cf. [1], [3]). *Let $\Gamma \neq \Delta$ be irreducible germs at $0 \in \mathbb{C}^2$ of curves with a common tangent $y = 0$. Let $(p_i; 0 \leq i \leq N)$ (resp. $(q_i; 0 \leq i \leq N_1)$) be the sequence of Puiseux exponents of Γ (resp. of Δ) with respect to the coordinates x, y . Let us denote $P_i = \text{g.c.d.}$*

(p_0, \dots, p_i) for $0 \leq i \leq N$ and $Q_i = g.c.d. (q_0, \dots, q_i)$ for $0 \leq i \leq N_1$. Then there exists an integer s , $0 \leq s \leq \min(N, N_1)$ such that

$$(2.1.1) \quad \frac{p_i}{p_0} = \frac{q_i}{q_0} \text{ for all } 0 \leq i \leq s.$$

(2.1.2) The intersection multiplicity $(\Gamma \cdot \Delta)$ of germs Γ, Δ is equal to the minimum of two numbers

$$\sum_{i=1}^s (P_{i-1} - P_i)q_i + P_s q_{s+1}, \quad \sum_{i=1}^s (Q_{i-1} - Q_i)p_i + Q_s p_{s+1}.$$

The reader can find a proof of Theorem 2.1 in the book [3]. Examples of calculations which lead to the above result are also presented in [1].

3. The proof of Theorem 1.1 is based on the

PROPOSITION 3.1. Let Γ, Δ, Λ be pairwise distinct irreducible germs of analytic curves at a point of a two-dimensional complex manifold. Suppose Λ is smooth and put $p = (\Gamma \cdot \Lambda)$, $q = (\Delta \cdot \Lambda)$, $(p, q) = g.c.d(p, q)$. Then

$$(\Gamma \cdot \Delta) \equiv 0 \left(\text{mod } \frac{p}{(p, q)} \text{ or } \frac{q}{(p, q)} \right).$$

Proof. We may assume without loss of generality that Γ, Δ are germs at $0 \in \mathbb{C}^2$ and that Λ is a germ of a line through the origin. If Γ, Δ have no common tangent then $\Gamma \cdot \Delta = \text{ord } \Gamma \cdot \text{ord } \Delta$ and $\text{ord } \Gamma = p$ or $\text{ord } \Delta = q$. Thus it suffices to consider the case where Γ, Δ have a common tangent Λ_0 . By a \mathbb{C} -linear transformation, we can arrange that Λ_0 has the equation $y = 0$. In the notation of 2.1 we have $(\Gamma \cdot \Lambda) = p_0$, $(\Delta \cdot \Lambda) = q_0$ if $\Lambda \neq \Lambda_0$ and $(\Gamma \cdot \Lambda_0) = p_1$, $(\Delta \cdot \Lambda_0) = q_1$. Therefore we have to check for $i = 0, 1$ the congruences

$$(a) \quad (\Gamma \cdot \Delta) \equiv 0 \left(\text{mod } \frac{p_i}{(p_i, q_i)} \text{ or } \frac{q_i}{(p_i, q_i)} \right).$$

If $s = 0$, then by (2.1.2) we have $(\Gamma \cdot \Delta) = \min(p_1 q_0, p_0 q_1)$ and (a) becomes evident. Now we assume that $s > 0$. Then we get from (2.1.2) the congruence

$$(b) \quad (\Gamma \cdot \Delta) \equiv 0 \pmod{(P_s \text{ or } Q_s)}.$$

On the other hand, we have for $0 \leq i \leq s$

$$(c) \quad P_s \equiv 0 \left(\text{mod } \frac{p_i}{(p_i, q_i)} \right) \text{ and } Q_s \equiv 0 \left(\text{mod } \frac{q_i}{(p_i, q_i)} \right).$$

Indeed, it follows from (2.1.1), by induction on i , that we have $\frac{P_i}{p_0} = \frac{Q_i}{q_0}$ for $0 \leq i \leq s$.

The equality $\frac{P_s}{p_0} = \frac{Q_s}{q_0}$ and the equalities (2.1.1) imply that $q_i P_s = p_i Q_s$ for $0 \leq i \leq s$.

This implies (c). Now (a) is an immediate consequence of (b) and (c) since $s > 0$. This concludes the proof of the proposition. We can now prove Theorem 1.1.

Proof of 1.1. If the curves C and D do not intersect at infinity then $k = mn$ and the theorem is obvious. Let us assume that C and D have a common point at infinity (necessarily one) and let Γ, Δ, A be the germs of \bar{C}, \bar{D}, L at the point. Obviously $(\Gamma \cdot A) = m$ and $(\Delta \cdot A) = n$ then by proposition (3.1) we obtain

$$(\Gamma \cdot \Delta) \equiv 0 \left(\text{mod } \frac{m}{(m, n)} \text{ or } \frac{n}{(m, n)} \right).$$

By classical Bezout's theorem we have $k = mn - (\Gamma \cdot \Delta)$ and our result is established.

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