UNIVERSITATIS IAGELLONICAE ACTA MATHEMATICA, FASCICULUS XXXVIII
2000

EFFECTIVE NULLSTELLENSATZ FOR STRICTLY REGULAR SEQUENCES

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Abstract. Sharp versions of the Nullsellensatz for a class of polynomial sequences are given.

1. Main results. Let $F_1, \dots, F_k : \mathbb{C}^n \to \mathbb{C}$, $1 \leqslant k \leqslant n$ be a sequence of nonconstant polynomials. Let $d_i = \deg F_i$ for $i = 1, \dots, k$.

DEFINITION 1.1. The sequence $F_1, \dots, F_k$ is strictly regular (s.r.) if there are linear forms $L_1, \dots, L_{n-k} : \mathbb{C}^n \to \mathbb{C}$ such that the mapping

$$(F_1, \dots, F_k, L_1, \dots, L_{n-k}) : \mathbb{C}^n \to \mathbb{C}^n$$

is proper.

REMARKS 1.2.

(1) If $k = 1$ then nonconstant F_1 is an s.r. sequence.
(2) If $k = n$ then $F_1, \dots, F_n$ is an s.r. sequence if and only if $(F_1, \dots, F_n) : \mathbb{C}^n \to \mathbb{C}^n$ is a proper mapping.
(3) Let $F_i^* : \mathbb{C}^n \to \mathbb{C}$ be a homogeneous form of degree d_i such that we have $F_i = F_i^* +$ (terms of degree $< d_i$) for $i = 1, \dots, k$. Suppose that the algebraic set $(F_1^*, \dots, F_k^*)^{-1}(0)$ is of dimension $n - k$, then $F_1, \dots, F_k$ is an s.r. sequence.
(4) If $F_1, \dots, F_k$ is an s.r. sequence then $(F_1, \dots, F_k)^{-1}(0)$ is of pure dimension $n - k$. In particular the algebraic cycle of zeroes $Z_{(F_1, \dots, F_k)}$ is defined (see [**7**]).

1991 *Mathematics Subject Classification.* 14E99, 32B99.
Key words and phrases. Proper polynomial mapping, geometric degree.
Work partially supported by KBN Grant 2 P03A 001 15.

THEOREM 1.3. *Let* $F_1, \dots, F_k : \mathbb{C}^n \to \mathbb{C}$ *be an s.r. sequence. If* $G : \mathbb{C}^n \to \mathbb{C}$ *is a nonconstant polynomial mapping such that* $G|(F_1, \dots, F_k)^{-1}(0) = 0$, *then there is a* $\mu > 0$ *such that*

$$G^\mu = A_1 F_1 + \ldots + A_k F_k,$$

with polynomials $A_1, \dots, A_k : \mathbb{C}^n \to \mathbb{C}$ *such that* $\deg(A_i F_i) \leqslant (\prod_{i=1}^{k} d_i)(\deg G)$ *for* $i = 1, \dots, k$

REMARKS 1.4.

(1) Theorem 1.3 is a generalization to the case of s.r. sequence of the main result of [**3**].
(2) From the given formulation of the theorem it follows immediately that $\mu \leqslant \prod_{i=1}^{k} d_i$.

THEOREM 1.5. *Let* $F_1, \dots, F_k : \mathbb{C}^n \to \mathbb{C}$ *be an s.r. sequence. If* $F_{k+1} : \mathbb{C}^n \to \mathbb{C}$ *is a polynomial of degree* $d_{k+1} > 0$ *such that* $(F_1, \dots, F_{k+1})^{-1}(0) = \emptyset$, *then*

$$A_1 F_1 + \ldots + A_{k+1} F_{k+1} = 1,$$

with polynomials $A_1, \dots, A_{k+1} : \mathbb{C}^n \to \mathbb{C}$ *such that* $\deg(A_i F_i) \leqslant d_1 \cdot \ldots \cdot d_{k+1}$ *for* $i = 1, \dots, k+1$

REMARKS 1.6.

(1) If degrees of polynomials in Theorem 1.5 are different than 2, then the theorem follows from Kollar's theorem (see [**5**]).
(2) If $k = n$ then [**3**] gives a better result.

2. Proofs. The proofs of Theorem 1.3 and Theorem 1.5 are based on the following proposition.

PROPOSITION 2.1. (cf. [**3**], [**6**]) *Suppose that* $H = (H_1, \dots, H_n) : \mathbb{C}^n \to \mathbb{C}^n$ *is a proper polynomial mapping of geometric degree* μ. *Then for every* $G : \mathbb{C}^n \to \mathbb{C}$ *of positive degree there exists a polynomial*

$$P_G(W, T) = T^\mu + P_G^{(1)}(W) T^{\mu-1} + \ldots + P_G^{(\mu)}(W),$$

in $n+1$ *variables* $(W, T) = (W_1, \dots, W_n, T)$ *such that:*

(1) $P_G(H_1, \dots, H_n, G) = 0$
(2) *For every* $w \in \mathbb{C}^n$ *we denote by* $\zeta^{(1)}(w), \dots, \zeta^{(\mu)}(w)$ *the sequence of all points of* $F^{-1}(0)$ *counted with multiplicities. Then*

$$P_G(w, T) = \prod_{j=1}^{\mu} (T - G(\zeta^{(j)}(w))).$$

(3) weight$(P_G) \leqslant (\prod_{i=1}^{n} d_i)(\deg G)$, *if* weight$(W_i) = \deg H_i = d_i$ *for* $i = 1, \ldots, n$ *and* weight$(T) = \deg G$.

Now let $F_1, \ldots, F_k : \mathbb{C}^n \to \mathbb{C}$ be an s.r. sequence of polynomials. Let us fix linear forms $L_1, \ldots, L_{n-k} : \mathbb{C}^n \to \mathbb{C}$ such that the mapping

$$H = (F_1, \ldots, F_k, L_1, \ldots, L_{n-k}) : \mathbb{C}^n \to \mathbb{C}^n.$$

is proper.

Proof of Theorem 1.3. Let $P(W,T) = P_G(W,T)$. From Prop. 2.1 (2) and from the assumption $G|(F_1, \ldots, F_k)^{-1}(0) = 0$ it follows that $P(0, w_{k+1}, \ldots, w_n, T) = T^\mu$ for every $(w_{k+1}, \ldots, w_n) \in \mathbb{C}^{n-k}$. This means that $P(0, W_{k+1}, \ldots, W_n, T) = T^\mu$, and consequently $P(W,T) = T^\mu + W_1 P_1(W,T) + \ldots + W_k P_k(W,T)$ with

$$\text{weight}(W_j P_j(W,T)) \leqslant \text{weight}\, P(W,T) \leqslant (\prod_{i=1}^{n} d_i)(\deg G) \text{ for } j = 1, \ldots, k\ .$$

Substitution $F_1, \ldots, F_k, G$ for $W_1, \ldots, W_k, T$ gives

$$G^\mu = (-P_1(F,L,G))F_1 + \ldots + (-P_k(F,L,G))F_k,$$

with

$$\deg(-P_i(F,L,G)F_i) \leqslant \text{weight}(-P_i(W,T)W_i) \leqslant (\prod_{i=1}^{k} d_i)(\deg G),$$

and the proof is complete. □

Proof of Theorem 1.5. Let $P(W,T) = P_G(W,T)$ and $G = F_{k+1}$. Since $P(H,G) = 0$ then we get

$$-(G^\mu + P^{(1)}(F)G^{\mu-1} + \cdots + P^{(\mu-1)}(F)G) = P^{(\mu)}(H).$$

This gives

$$(*) \qquad\qquad - F_{k+1}A_{k+1} = P^{(\mu)}(H)$$

We have $P^{(\mu)}(w) = \pm \prod_{j=1}^{\mu} F_{k+1}(\zeta^{(j)}(w))$, consequently $P^{(\mu)}(0, w_1, \ldots, w_k) \neq 0$ in $\mathbb{C}^{n-k}$ and we may assume

$$P^{(\mu)}(0, w_{k+1}, \ldots, w_n) = -1\ .$$

Then $P^{(\mu)}(W) = W_1 A_1(W) + \ldots + W_k A_k(W) - 1$ with

$$\text{weight}(W_k A_k) \leqslant \text{weight}(P^{(\mu)}) \leqslant d_1 \cdot \ldots \cdot d_k\ .$$

Substitution gives $P^{(\mu)}(H) = F_1 A_1(F) + \ldots + F_k A_k(F) - 1$ with $\deg(F_i A_i(F)) \leqslant d_1 \cdot \ldots \cdot d_{k+1}$, for $i = 1, \ldots, k$. By $(*)$ we get $A_1 F_1 + \ldots + A_{k+1} F_{k+1} = 1$ with $\deg(F_i A_i(F)) \leqslant d_1 \cdot \ldots \cdot d_{k+1}$, for $i = 1, \ldots, k+1$. This ends the proof. □

3. Examples. Let $F = (F_1, \dots, F_k) : \mathbb{C}^n \to \mathbb{C}^k$ be a polynomial mapping. One can compare properties of the mapping F with the condition that $F_1, \dots, F_k$ is an s.r. sequence.

It is easy to see that if $F_1, \dots, F_k$ is an s.r. sequence then the mapping F is surjective and has all fibres of pure dimension $n - k$. In the next example we observe that this assumption with additional complete intersection of components of F is not enough for $F_1, \dots, F_k$ to be an s.r. sequence.

EXAMPLE 3.1. Consider polynomials:

$$F_1 : \mathbb{C}^3 \ni (x, y, z) \to x \in \mathbb{C},$$
$$F_2 : \mathbb{C}^3 \ni (x, y, z) \to xy^2 + y \in \mathbb{C}.$$

It is easy to see that F_1, F_2 is a complete intersection and every fibre of F has dimension 1. We observe that F_1, F_2 is not an s.r.sequence. Let us take a linear form $L(x, y, z) = ax + by + cz$ and suppose that the mapping $H = (F_1, F_2, L)$ is proper. Then obviously $c \neq 0$ and we can assume $c = 1$. Taking the sequence $h_\nu = (\nu^{-1}, -\nu, b\nu - a\nu^{-1})$, we have $|h_\nu| \to +\infty$ $(\nu \to +\infty)$. Observe that we obtain $H(h_\nu) = (\nu^{-1}, 0, 0) \to 0$ $(\nu \to +\infty)$, and so (F_1, F_2) is not an s.r. sequence.

Our next example says that an additional condition that all the fibres have the same degree (with multiplicity) is too strong in this situation.

EXAMPLE 3.2. Consider polynomials:

$$F_1 : \mathbb{C}^3 \ni (x, y, z) \to y \in \mathbb{C},$$
$$F_2 : \mathbb{C}^3 \ni (x, y, z) \to z + x^2 y \in \mathbb{C}.$$

It is easy see that F_1, F_2 is an s.r. sequence since the mapping (F_1, F_2, L) is a polynomial automorphism for $L(x, y, z) = x$. But $\deg F^{-1}((0,0)) = 1$ and $\deg F^{-1}((1,0)) = 2$.

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Received January 17, 2000

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