An Inequality for the Intersection multiplicity of Analytic Curves

by

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Summary. Let $f = f_1 \ldots f_k$ and $g = g_1 \ldots g_l$ be decompositions of the power series f, $g \in C\{x, y\}$ into irreducible factors. Then, for each irreducible power series $h \in C\{x, y\}$, we have

$$m_0(f, h)/\operatorname{ord} h \leqq \max_{j=1}^{l} \{m_0(f, g_j)/\operatorname{ord} g_j\}$$

or

$$m_0(g, h)/\operatorname{ord} h \leqq \max_{i=1}^{k} \{m_0(f_i, g)/\operatorname{ord} f_i\}.$$

Here $m_0(f, g)$ denotes the intersection multiplicity of the analytic curves $\{f = 0\}$, $\{g = 0\}$, and ord h stands for the order of h.

1. Inequality for the intersection multiplicity.

Let $C\{x, y\}$ be the ring of convergent power series in variables x, y with complex coefficients. In what follows we assume that f, $g \in C\{x, y\}$ are non-zero power series without constant terms. We denote by $m_0(f, g)$ the intersection multiplicity of the analytic curves $\{f = 0\}$ and $\{g = 0\}$.

Recall that the following conditions are equivalent:

(i) $m_0(f, g) < +\infty$ (we adopt the usual conventions on the symbol $+\infty$),

(ii) the series f, g have not a non-invertible common divisor in $C\{x, y\}$,

(iii) the system of equations $f = 0$, $g = 0$ has, near the origin of C^2, only the solution $(0, 0)$.

All the properties of the intersection multiplicity we need are proved in [4] (pp. 45–51).

We use ord h to denote the order of the series $h \in C\{x, y\}$.

The aim of this note is the following

(1.1) THEOREM. *Let* $f = f_1 \ldots f_k$ *and* $g = g_1 \ldots g_l$ *be decompositions of the series* f, g *into irreducible factors. Then, for every irreducible power series* h, *we have*

$$m_0(f, h)/\text{ord } h \leq \max_{j=1}^{l} \{m_0(f, g_j)/\text{ord } g_j\}$$

or

$$m_0(g, h)/\text{ord } h \leq \max_{i=1}^{k} \{m_0(f_i, g)/\text{ord } f_i\}.$$

The proof of the above Theorem is given in Section 3 of this note. Now, we present some corollaries.

(1.2) COROLLARY. *Let* f, g, $h \in \mathbf{C}\{x, y\}$. *Suppose that* h *is irreducible. Then* $m_0(f, g) \geq \min \{m_0(f, h)/\text{ord } h, m_0(g, h)/\text{ord } h\}$.

Proof. Let $f = f_1 \ldots f_k$ and $g = g_1 \ldots g_l$ be decompositions of f and g into irreducible factors. Obviously, we have $m_0(f, g_j)/\text{ord } g_j \leq m_0(f, g)$ for all $j = 1, \ldots, l$ and $m_0(f_1, g)/\text{ord } f_i \leq m_0(f, g)$ for all $i = 1, \ldots, k$. According to Theorem 1.1, these inequalities imply the corollary.

The inequality due to Namba (cf. [3], p. 74): $m_0(f, g) \geq \min \{m_0(f, h), m_0(g, h)\}$ with ord $h = 1$ is a special case of Corollary 1.2 since every power series of order 1 is irreducible.

(1.3) COROLLARY (cf. [5]). *Let* f, g, $h \in \mathbf{C}\{x, y\}$ *be irreducible power series. Then at least two of the three numbers*

$$m_0(f, g)/\text{ord } f \text{ ord } g, \quad m_0(f, h)/\text{ord } f \text{ ord } h, \quad m_0(g, h)/\text{ord } g \text{ ord } h$$

are equal and the third is not smaller than the other two.

Proof. We may assume that the three numbers listed above form an increasing sequence. From Theorem 1.1 we get $m_0(f, h)/\text{ord } f \text{ ord } h \leq m_0(f, g)/\text{ord } f \text{ ord } g$ or $m_0(g, h)/\text{ord } g \text{ ord } h \leq m_0(f, g)/\text{ord } f \text{ ord } g$, hence $m_0(f, g)/\text{ord } f \text{ ord } g = m_0(f, h)/\text{ord } f \text{ ord } h \leq m_0(g, h)/\text{ord } g \text{ ord } h$ and the corollary is proved.

In order to present another application of our result, let us recall that the Łojasiewicz exponent $l_0(f, g)$ is defined by the formula $l_0(f, g) = \inf \{\theta > 0 : \max\{|f(x, y)|, |g(x, y)|\} \geq C \, [\max\{|x|, |y|\}]^\theta$ near the origin of $\mathbf{C}^2\}$. The basic properties of $l_0(f, g)$ are given in [2], [1] and [6]. Note that $l_0(f, g) < +\infty$ if and only if $m_0(f, g) < +\infty$. In [1] an explicit formula (generalizing a result proved in [5]) for $l_0(f, g)$ was given. Theorem 1.1 allows us to give a short proof of that formula.

(1.4) COROLLARY (cf. [1], [5]). *With the notations as above,*

$$(1) \qquad l_0(f, g) = \max\{\max_{i=1}^{k} \{m_0(f_i, g)/\text{ord } f_i\}, \max_{j=1}^{l} \{m_0(f, g_j)/\text{ord } g_j\}\}.$$

Proof. According to the characterization of $l_0(f, g)$ in terms of analytic arcs (cf. [2], [6]), we may write

(2) $\quad l_0(f, g) = \sup\{\min\{m_0(f, h)/\operatorname{ord} h, \ m_0(g, h)/\operatorname{ord} h\}:h-\text{irreducible}\}$.

Let $l^* =$ the right side of the equality stated in (1). From (2) and Theorem 1.1 we immediately get the estimate $l_0(f, g) \leqq l^*$. On the other hand, we have, by (2), $l_0(f, g) \geqq \min\{m_0(f, h)/\operatorname{ord} h, \ m_0(g, h)/\operatorname{ord} h\}$ for any irreducible $h \in C\{x, y\}$. Substituting for h the series f_i $(i = 1, \ldots, k)$ and g_j $(j = 1, \ldots, l)$, we obtain the inequality $l_0(f, g) \geqq l^*$. This proves the corollary.

2. Order of contact.

Let A be a non-empty set. A function $d:A \times A \to \mathbf{R} \cup \{+\infty\}$ satisfying, for arbitrary $a, b, c \in A$, the conditions

(i) $d(a, a) = +\infty$,

(ii) $d(a, b) = d(b, a)$,

(iii) $d(a, b) \geqq \min\{d(a, c), d(b, c)\}$

will be called an order of contact in A.

One easily checks

(2.1) LEMMA. *Condition* (iii) *is equivalent to the following: at least two of the numbers* $d(a, b), d(a, c), d(b, c)$ *are equal and the third is not smaller than the other two.*

The proposition given below will be useful in the proof of Theorem 1.1.

(2.2) PROPOSITION. *Let d be an order of contact in A. For any* $a_1, \ldots, a_m$, $b_1, \ldots, b_n$, $c \in A$, *at least one of the following conditions holds:*

(I) *there exists an* $s \in \{1, \ldots, n\}$ *such that, for any* $r \in \{1, \ldots, m\}$, $d(a_r, c) \leqslant d(a_r, b_s)$,

(II) *there exists an* $r \in \{1, \ldots, m\}$ *such that, for any* $s \in \{1, \ldots, n\}$, $d(b_s, c) \leqq d(a_r, b_s)$.

Proof. Let us suppose that neither (I) nor (II) holds. Then, for any $s \in \{1, \ldots, n\}$, there exists an index $\varrho(s) \in \{1, \ldots, m\}$ such that $d(a_{\varrho(s)}, c) > d(a_{\varrho(s)}, b_s)$ and, for any $r \in \{1, \ldots, m\}$, there exists $\sigma(r) \in \{1, \ldots, n\}$ such that $d(b_{\sigma(r)}, c) > d(a_r, b_{\sigma(r)})$. Applying Lemma 2.1 to $a_{\varrho(s)}$, b_s, c and to a_r, $b_{\sigma(r)}$, c, we get

(1) $$d(a_{\varrho(s)}, b_s) = d(b_s, c) < d(a_{\varrho(s)}, c),$$

(2) $$d(a_r, b_{\sigma(r)}) = d(a_r, c) < d(b_{\sigma(r)}, c).$$

We may assume without loss of generality that

(3) $$d(a_{\varrho(1)}, b_1) = \max_{s=1}^{n} \{d(a_{\varrho(s)}, b_s)\}.$$

Using successively (1), (2) and again (1), we get $d(a_{\varrho(1)}, b_1) < d(a_{\varrho(1)}, c) = d(a_{\varrho(1)}, b_{\sigma(\varrho(1))}) < d(b_{\sigma(\varrho(1))}, c) = d(a_{\varrho(s_1)}, b_{s_1})$ with $s_1 = \sigma(\varrho(1))$. Thus we have $d(a_{\varrho(1)}, b_1) < d(a_{\varrho(s_1)}, b_{s_1})$, which contradicts assumption (3). This ends the proof of the proposition.

3. Proof of the main result. Let $m = \mathrm{ord}\ f$, $n = \mathrm{ord}\ g$, $p = \mathrm{ord}\ h$. Using a linear change of coordinates and the Weierstrass Preparation Theorem, we can assume that f, g, h are y-distinguished polynomials of degress m, n, p, respectively. According to the Newton–Puiseux Theorem, there is an integer $D > 0$ such that

$$f(T^D, y) = \prod_{r=1}^{m} (y - a_r(T)), \quad g(T^D, y) = \prod_{s=1}^{n} (y - b_s(T)),$$

$$h(T^D, y) = \prod_{t=1}^{p} (y - c_t(T))$$

in the ring of polynomials with coefficients in the ring $\mathbf{C}\{T\}$. Using the well-known properties of the intersection multiplicity (cf. [1], [4]), we get that, for each $t \in \{1, \ldots, p\}$,

$$(1) \qquad \sum_{r=1}^{m} \mathrm{ord}(c_t - a_r) = Dm_0(f, h)/\mathrm{ord}\ h,$$

$$(2) \qquad \sum_{s=1}^{n} \mathrm{ord}(c_t - b_s) = Dm_0(g, h)/\mathrm{ord}\ h.$$

Moreover, for each $s \in \{1, \ldots, n\}$, there exists a $j \in \{1, \ldots, l\}$ such that

$$(3) \qquad \sum_{r=1}^{m} \mathrm{ord}(b_s - a_r) = Dm_0(f, g_j)/\mathrm{ord}\ g_j$$

and, for each $r \in \{1, \ldots, m\}$, there exists an $i \in \{1, \ldots, k\}$ such that

$$(4) \qquad \sum_{s=1}^{n} \mathrm{ord}(a_r - b_s) = Dm_0(f_i, g)/\mathrm{ord}\ f_i.$$

The function $d: \mathbf{C}\{T\} \times \mathbf{C}\{T\} \to \mathbf{R} \cup \{+\infty\}$ given by $d(a, b) = \mathrm{ord}\ (a - b)$ is an order of contact in $\mathbf{C}\{T\}$. Hence from Proposition 2.2 it follows that, for any $t \in \{1, \ldots, p\}$, we have

$$\sum_{r=1}^{m} \mathrm{ord}(c_t - a_r) \leq \sum_{r=1}^{m} \mathrm{ord}(b_s - a_r)$$

for some $s \in \{1, \ldots, n\}$, or

$$\sum_{s=1}^{n} \mathrm{ord}(c_t - b_s) \leq \sum_{s=1}^{n} \mathrm{ord}(a_r - b_s)$$

for some $r \in \{1, \ldots, m\}$. Using (1), (2), (3) and (4), we get

$$Dm_0(f, h)/\mathrm{ord}\ h \leq Dm_0(f, g_j)/\mathrm{ord}\ g_j \text{ for some } j \in \{1, \ldots, l\}$$

or

$$Dm_0(g, h)/\mathrm{ord}\ h \leq Dm_0(f_i, g)/\mathrm{ord}\ f_i \text{ for some } i \in \{1, \ldots, k\},$$

and the theorem follows.

4. Concluding remarks. A slight modification of the proof given above shows that we may generalize Theorem 1.1 by replacing ord h, ord g_j, ord f_i by $m_0(h, L)$, $m_0(g_j, L)$, $m_0(f_i, L)$ provided L is a linear form not dividing any of the series h, g_j, f_i. In fact, using a linear change of coordinates and the Weierstrass Preparation Theorem, we may assume that $L(x, y) = x$, and that f, g, h are y-distinguished polynomials of degrees $m = \text{ord } f(0, y) = m_0(f, L)$, $n = \text{ord }$ $g(0, y) = m_0(g, L)$ and $p = \text{ord } h(0, y) = m_0(h, L)$, respectively. Then the proof runs analogously to that of Theorem 1.1 provided we replace ord h, ord g_j, ord f_i by $m_0(h, L)$, $m_0(g_j, L)$, $m_0(f_i, L)$, respectively.

From Corollary 1.3 it follows that the function $(f, g) \mapsto m_0(f, g)/\text{ord } f$ ord g is an order of contact in the set of all irreducible power series. Applying Proposition 2.2, we get another strengthened version of Theorem 1.1.

All results proved in this note remain true it we replace $\mathbf{C}\{x, y\}$ by the ring of formal power series $\mathbf{C}[[x, y]]$.

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Я. Хондзыньски, А. Плоски, **Неравенства для кратности пересечения аналитических кривых**

Пусть $f = f_1, \dots f_k$, $g = g_1 \dots g_l$ разложения рядов $f, g \in \mathbf{C}\{x, y\}$ на неприводимые элементы. Если h — неприводимый элемент кольца $\mathbf{C}\{x, y\}$, то

$$m_0(f, h)/\text{ord } h \leq \max_{j=1}^{l} \{m_0(f, g_j)/\text{ord } g_j\}$$

или

$$m_0(g, h)/\text{ord } h \leq \max_{i=1}^{k} \{m_0(f_i, g)/\text{ord } f_i\}.$$

Здесь $m_0(f, g)$ — кратность пересечения аналитических кривых $\{f = 0\}$, $\{g = 0\}$ и ord h — порядок ряда h.