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FIELD THEORY AND POLYNOMIALS

On the Irreducibility of Polynomials in Several Complex Variables

by

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Summary. We give a criterion of irreducibility for polynomials of $n > 2$ variables and show how to ensure the connectedness of zeros of a polynomial of two variables. Using Bertini's Theorem we prove that for any polynomial f there is an irreducible polynomial f_0 such that $f \in \mathbf{C}[f_0]$.

1. Introduction. Let $f = f(x)$ be a nonconstant polynomial in n complex variables $x = (x_1, \ldots, x_n)$. Write $f = \sum_{k=0}^{m} f_k$ where f_k is a homogeneous polynomial of degree k. We say that f has no singularities at infinity if the system of homogeneous equations

$$\frac{\partial f_m}{\partial x_1} = \ldots = \frac{\partial f_m}{\partial x_n} = f_{m-1} = 0$$

has no solutions in $\mathbf{C}^n - \{0\}$.

THEOREM 1. *Let f be a polynomial in n variables which has no singularities at infinity. If $n > 2$, then f is irreducible. If $n = 2$, then f is nearly irreducible, i.e. any two nonconstant polynomial factors of f have a common zero in $\mathbf{C}^2$.*

The proof of Theorem 1 is given in Section 2. Recall here that every nearly irreducible polynomial has connected its zero-set, but need not be irreducible (cf. [1]).

REMARK. If $f = f(x, y)$ is nearly irreducible and $\operatorname{grad} f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) \neq 0$ on $f(x, y) = 0$ then f is irreducible (cf. [6], Corollary).

Let us assume that $p_1, \ldots, p_n > 0$ are integers. A polynomial f will be called $(p_1,\dots,p_n)$-*polynomial* if

$$f = a_1x_1^{p_1} + \dots + a_nx_n^{p_n} + \sum c_{i_1,\dots,i_n}x_1^{i_1}\dots x_n^{i_n}, \qquad a_1,\dots,a_n \neq 0$$

where the summation is taken over all sequences $(i_1,\dots,i_n)$ such that $i_1/p_1 + \dots + i_n/p_n < 1$.

Any polynomial of the form $\sum_{i=1}^{n} P_i(x_i)$ with nonconstant $P_i(x_i)$ is a $(\deg P_1,\dots,\deg P_n)$-polynomial. The generalized difference polynomials defined in [1] are (p_1,p_2)-polynomials. The second part of the following corollary is due to [1, 6].

COROLLARY to Theorem 1. *If $n > 2$ then any $(p_1,\dots,p_n)$-polynomial is irreducible. Any (p_1,p_2)-polynomial is nearly irreducible.*

Proof. Let f be a $(p_1,\dots,p_n)$-polynomial. Take $p = p_1\dots p_n$ and put $\overline{f}(x_1,\dots,x_n) = f(x_1^{p/p_1},\dots,x_n^{p/p_n})$. Then $\overline{f}(x_1,\dots,x_n) = a_1x_1^p + \dots + a_nx_n^p +$ (terms of degree $< p$), so by Theorem 1 the polynomial $\overline{f}$ is irreducible if $n > 2$ and nearly irreducible if $n = 2$. Obviously the same is true for f.

REMARK. Any (p_1,p_2)-polynomial has at most $\mathrm{GCD}(p_1,p_2)$ factors. In particular, if p_1,p_2 are relatively prime than any (p_1,p_2)-polynomial is irreducible (cf. [3]).

Let $f = f(x,y) = \sum c_{\alpha,\beta}x^\alpha y^\beta$ be a polynomial of two variables. Recall that the Newton polygon $N(f)$ of f is the convex hull of the set $\{(0,0)\} \cup \{(\alpha,\beta) \in \mathbb{N}^2 : c_{\alpha,\beta} \neq 0\}$. We denote by $\partial N(f)$ the set of all segments of the boundary of $N(f)$ which do not lie on the axes $\alpha = 0$, $\beta = 0$. With any $S \in \partial N(f)$ we associate a polynomial $f_S(x,y) = \sum_{(\alpha,\beta)\in S} c_{\alpha,\beta}x^\alpha y^\beta$. We say that f is non-degenerate if for any $S \in \partial N(f)$ the system of equations $\frac{\partial f_S}{\partial x} = \frac{\partial f_S}{\partial y} = 0$ has no solution in $(\mathbb{C}^2 - \{0\}) \times (\mathbb{C}^2 - \{0\})$. If for every $S \in \partial N(f)$ the set $S \cap \{(\alpha,\beta) \in \mathbb{N}^2 : c_{\alpha,\beta} \neq 0\}$ contains only the ends of S then f is non-degenerate.

THEOREM 2. *Let $f = f(x,y)$ be a non-degenerate polynomial of two variables. Suppose that:*

1) *$N(f)$ intersects the axes $\alpha = 0$ and $\beta = 0$ at points different from $(0,0)$.*

2) *every segment $S \in \partial N(f)$ has a negative slope, i.e. S lies on the line $\alpha + \mu\beta = \nu$ with $\mu > 0$.*

Then f is nearly irreducible.

Proof of Theorem 2 is given in Section 3. Theorem 2 is a generalization of the result of [1]. Imdeed, if $f = f(x,y)$ is a (p,q)-polynomial and S is the segment joining the points $(p,0)$ and $(0,q)$ then $N(f) = \{S\}$ and $f_S(x,y) = x^p + y^p$, therefore the assumptions of Theorem 2 are satisfied.

A nonconstant polynomial f will be called *primitive* if the polynomials $f - c$ are irreducible for all but finite number of $c \in \mathbf{C}^n$. Let t be a variable. From Noether's Theorem (cf. [7] p. 71) it follows easily that $f = f(x)$ is primitive if and only if the polynomial $f(x) - t$ is irreducible in $\overline{\mathbf{C}(t)}[x]$ where $\overline{\mathbf{C}(t)}$ is the algebraic closure of the field $\mathbf{C}(t)$. Moreover, for any nonconstant polynomial there are two possibilities: f is primitive or $f - c$ is reducible for all $c \in \mathbf{C}$. In particular any irreducible polynomial is primitive. The theorem presented below is the n-dimensional generalization of a result proved by ([4], p. 100).

THEOREM 3. *For any polynomial $f = f(x)$ there exists a primitive polynomial $f_0 = f_0(x)$ and a polynomial of one variable $P = P(t)$ such that $f = P(f_0)$.*

Proof of Theorem 3 based on Bertini's theorem is given in Section 4 of this paper. Note that the polynomial f_0 in Theorem 3 can be chosen to be irreducible (replace f_0 by $f_0 - c$ and $P(t)$ by $P(t + c)$).

2. Proof of Theorem 1. The following lemma is well known and easy to prove by using the resultant

LEMMA 1. *Any two homogeneous polynomials of $n > 2$ variables have a common zero in $\mathbf{C}^n - \{0\}$.*

Let $n > 2$. Suppose that the polynomial f of n variables has no singularities at infinity. We have to show that f is irreducible. To get the contradiction suppose that $f = gh$ with nonconstant polynomials g, h. Write

$$f = f_m + f_{m-1} + \dots, \qquad g = g_k + g_{k-1} + \dots, \qquad h = h_l + h_{l-1} + \dots$$

where $f_m, \dots, g_k, \dots, h_l, \dots$ are homogeneous polynomials.

The equality $f = gh$ implies

$$f_m = g_k h_l\,, \tag{1}$$

$$f_{m-1} = g_k h_{l-1} + g_{k-1} h_l\,. \tag{2}$$

Differentiating (1) yields

$$\frac{\partial f_m}{\partial x_i} = \frac{\partial g_k}{\partial x_i} h_l + g_k \frac{\partial h_l}{\partial x_i} \quad \text{for } i = 1, \dots, n\,. \tag{3}$$

By Lemma 1 there is an $a \in \mathbf{C}^n - \{0\}$ such that $g_k(a) = h_l(a) = 0$, therefore $\frac{\partial f_m}{\partial x_1}(a) = \dots = \frac{\partial f_m}{\partial x_n}(a) = f_{m-1}(a) = 0$ by (3) and (2), which contradicts the assumption of the Theorem 1.

Let $n = 2$. Suppose that $f = f(x_1, x_2)$ has no singularities at infinity. Let g, h be two nonconstant polynomial factors of f. We have to show that the equations $g = 0$, $h = 0$ have a common solution in $\mathbb{C}^2$. Let us write $f = ghP$ and

$$f = f_m + f_{m-1} + \dots, \qquad g = g_k + g_{k-1} + \dots,$$
$$h = h_l + h_{l-1} + \dots, \qquad P = P_s + P_{s-1} + \dots$$

with homogeneous $f_m, g_k, h_l, P_s, \dots$ It is sufficient to show that the homogeneous equations $g_k = 0$, $h_l = 0$ have no solutions in $\mathbb{C}^2 - \{(0,0)\}$. In fact, if $g_k(x_1, x_2) = h_l(x_1, x_2) = 0$ implies $x_1 = x_2 = 0$, then the system $g(x_1, x_2) = h(x_1, x_2) = 0$ has a solution in $\mathbb{C}^2$ (we apply Lemma 1 to the homogenizations of polynomials g, h).

Now, let $a = (a_1, a_2) \in \mathbb{C}^2$ be such that $g_k(a) = h_l(a) = 0$. We will show that $a = (0,0)$. From $f = ghP$ we get

$$f_m = g_k h_l P_s \tag{4}$$

$$f_{m-1} = g_{k-1} h_l P_s + g_k h_{l-1} P_s + g_k h_l P_{s-1} \tag{5}$$

Differentiating (4) gives

$$\frac{\partial f_m}{\partial x_i} = \frac{\partial g_k}{\partial x_i} h_l P_s + \frac{\partial h_l}{\partial x_i} g_k P_s + \frac{\partial P_s}{\partial x_i} g_k h_l \tag{6}$$

From (5) and (6) we get $\frac{\partial f_m}{\partial x_1}(a) = \frac{\partial f_m}{\partial x_2}(a) = f_{m-1}(a) = 0$, which implies $a = 0$ because f has no singularities at infinity.

3. Proof of Theorem 2. We follow the ideas of [1]. Let $\mathbb{C}((x^{-1}))^*$ be the field of Puiseux series of the form $y(x) = ax^r + a_1 x^{r_1} + \dots$ where $a, a_1, \dots \in \mathbb{C} - \{0\}$ and $r > r_1 > \dots$ is a decreasing sequence of rational numbers with a common denominator. We put $\deg y(x) = r$, $y^+(x) = ax^r$ and use the convention: $\deg 0 = -\infty$. The field $\mathbb{C}((x^{-1}))^*$ is an algebraically closed extension of $\mathbb{C}(x)$ and deg is an extension of the degree defined in $\mathbb{C}(x)$ (cf. [5] and [8] where the field $\mathbb{C}((x))^*$ isomorphic to $\mathbb{C}((x^{-1}))^*$ is considered). The set of elements of $\mathbb{C}((x^{-1}))^*$ of non-positive degree is a domain $\mathbb{C}[[x^{-1}]]^*$. If $y(x) \in \mathbb{C}[[x^{-1}]]^*$, the coefficient of x^0 in $y(x)$ will be designated as $y(\infty)$. Obviously $y(\infty) = 0$ if and only if $\deg y(x) < 0$.

Now we use the Implicit Function Theorem for Puiseux Series (cf. [5], p. 102). Let $F(x, Y) \in \mathbb{C}[[x^{-1}]]^*[Y]$ be a polynomial in one variable Y. If $a \in \mathbb{C}$ is a simple root of the polynomial $F(\infty, Y) \in \mathbb{C}[Y]$ then there exists unique $Y(x) \in \mathbb{C}[[x^{-1}]]^*$ such that $F(x, Y(x)) = 0$ and $Y(\infty) = a$.

The Newton method of determining the solutions of $f(x,y) = 0$ in $\mathbb{C}((x^{-1}))^*$ and the Implicit Function Theorem (IFT) give

LEMMA 2. *Let $f(x,y) \in \mathbb{C}[x,y]$, $\deg f(x,0) > 0$ and $y(x) \in \mathbb{C}((x^{-1}))^*$. If $f(x, y(x)) = 0$ then there is a segment $S \in \partial N(f)$ such that $f_S(x, y^+(x)) = 0$.*

If $\frac{\partial f_S}{\partial y}(x, y^+(x)) \neq 0$ *then the solution* $y(x)$ *of the equation* $f(x,y) = 0$ *is uniquely determined by* $y^+(x)$.

Proof. Let $f(x,y) = \sum c_{\alpha,\beta} x^\alpha y^\beta$, $A = \{c_{\alpha,\beta} \in \mathbb{N}^2 : c_{\alpha,\beta} \neq 0\}$. The classical reasoning (cf. [8] p. 98) shows that there is a line with equation $\alpha + \mu\beta = \nu$ ($\mu = \deg y(x)$) such that

(i) if $(\alpha,\beta) \in A$ then $\alpha + \mu\beta \leqslant \nu$,

(ii) the set $A_0 = \{(\alpha,\beta) \in \mathbb{N}^2 : \alpha + \mu\beta = \nu\}$ contains at least two points,

(iii) $\sum_{(\alpha,\beta)\in A_0} c_{\alpha,\beta} x^\alpha (y^+(x))^\beta = 0$.

Let $\nu_0 = \deg f(x,0)$. Then $(0,\nu_0) \in A$ and $\nu \leqslant \nu_0 > 0$ by (i). Therefore $\alpha + \mu\beta = \nu$ is a supporting line of the convex set $N(f)$, different from axes $\alpha = 0$, $\beta = 0$. Consequently, there is a segment $S \in \partial N(f)$ such that $S \cap A = A_0$. We get $f_S(x, y^+(x)) = 0$ by (iii).

To check the second part of Lemma 2 let $y(x) = ax^\mu + \dots$, $Y(x) = x^{-\mu} y(x)$. We have $f(x, x^\mu Y) = x^\nu F(x,Y)$ where $F(x,Y) \in \mathbb{C}[[x^{-1}]]^*[Y]$. It is easy to see that $Y(\infty) = a$ is a simple root of $F(\infty, Y) = f_S(1,Y)$ whence by IFT the $Y(x)$ and consequently $y(x) = x^\mu Y(x)$ are uniquely determined.

On account of Lemma 2 let us note that if S lies on the line $\alpha + \mu\beta = \nu$ then any solution of $f_S(x,y) = 0$ is of the form ax^μ with $a \in \mathbb{C}$, whence $\deg y(x) = \deg y^+(x) = \mu$. If f is non-degenerate, then $\frac{\partial f_S}{\partial y}(x, ax^\mu) \neq 0$.

LEMMA 3. *Let* $f = f(x,y)$ *be a polynomial of two variables satisfying the assumptions of Theorem* 2. *Then*

$$(*) \qquad f(x,y) = a \prod_{i=1}^{m} (y - y_i(x)), \ a \in \mathbb{C} \quad \text{in } \mathbb{C}((x^{-1}))^*[y]$$

with $\deg y_i(x) > 0$ *for* $i = 1, \dots, m$. *Moreover, if* $y_i(x) \neq y_j(x)$ *then* $y_i^+(x) \neq y_j^+(x)$.

Proof. From assumptions on $N(f)$ it follows that

(1) $N(f)$ intersects the axis $\alpha = 0$ at point $(0,m) = (0, \deg_y f)$

(2) if $S, T \in \partial N(f)$ and $S \neq T$ then the slopes of S and T are different.

We can write $(*)$ by (1), because the field $\mathbb{C}((x^{-1}))^*$ is algebraically closed. The slopes of the segments $S \in \partial N(f)$ are negative, therefore $\deg y_i(x) > 0$ by the first part of Lemma 2. Suppose that $y_i(x) \neq y_j(x)$ and let $S, T \in \partial N(f)$ be such that $f_S(x, y_i^+(x)) = 0$, $f_T(x, y_j^+(x)) = 0$. If $S \neq T$ then $y_i^+(x) \neq y_j^+(x)$ since $\deg y_i^+(x) \neq \deg y_j^+(x)$ by (2). If $S = T$ then $y_i^+(x) \neq y_j^+(x)$ by the second part of Lemma 2, for f is non-degenerate.

To prove Theorem 2 let us consider two polynomial nonconstant factors $g(x,y)$, $h(x,y)$ of $f(x,y)$. With the notation of Lemma 3 we write $g(x,y) =$

$b\prod_{i\in I}(y-y_i(x))$, $h(x,y)=c\prod_{j\in J}(y-y_j(x))$, $b,c\in\mathbf{C}-\{0\}$. Let $R(x)=y$-resultant of $g(x,y)$, $h(x,y)$. The system of equations $g(x,y)=0$, $h(x,y)=0$, has a solution in $\mathbf{C}^2$ if and only if $R(x)\equiv 0$ or $\deg R(x)>0$. Assume $R(x)\not\equiv 0$, hence we get $R(x)=\text{const}\prod(y_i(x)-y_j(x))$ and by lemma 2

$$\deg R(x)=\sum_{i,j}\deg(y_i(x)-y_j(x))=\sum_{i,j}\max(\deg y_i(x),\deg y_j(x))>0\,,$$

since $\deg(y_i(x)-y_j(x))=\max(\deg(y_i(x),y_j(x))$ if $y_i^+(x)\neq y_j^+(x)$.

4. **Proof of Theorem 3.** The following Lemma is due to Gordan (cf. [2] and [5] p. 9–10 for the proof).

LEMMA 4. *Suppose that the polynomials $g=g(x)$ and $h=h(x)$ are algebraically dependent. Then there exists a polynomial $t(x)$ such that $g(x),h(x)\in\mathbf{C}[t(x)]$.*

To prove Theorem 3 it is sufficient to check that for any $f\in\mathbf{C}[x]$ which is not primitive there is a polynomial $f_1\in\mathbf{C}[x]$ such that $f\in\mathbf{C}[f_1]$ and $\deg f_1<\deg f$. Let $f=f(x)$ be a nonconstant polynomial which is not primitive and let t be a variable. It is easy to see that the polynomial $f(x)-t$ is irreducible in $\mathbf{C}(t)[x]$. By Bertini's Theorem (cf. [7], p. 79) there are polynomials $g=g(x)$, $h=h(x)$ such that

(1) $f(x)-t=a_0(t)g(x)^p+a_1(t)g(x)^{p-1}h(x)+\ldots+a_p(t)h(x)^p$ in $\mathbf{C}[t,x]$ and

(2) $\max(\deg g,\deg h)<\deg f$.

From (1) we get

(3) $f(x)=a_0(0)g(x)^p+a_1(0)g(x)^{p-1}h(x)+\ldots+a_p(0)h(x)^p$

(4) $-1=a_0'(0)g(x)^p+a_1'(0)g(x)^{p-1}h(x)+\ldots+a_p'(0)h(x)^p$

By (4) the polynomials $g(x)$, $h(x)$ are algebraically dependent. Therefore by Gordan's Lemma there is a polynomial $f_0=f_0(x)$ such that

(5) $g(x),h(x)\in\mathbf{C}[f_0(x)]$

Obviously $\deg f_0\leqslant\deg g,\deg h$ whence by (2): $\deg f_0<\deg f$. From (5) and (3) we get $f\in\mathbf{C}[f_0]$.

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