

On the Jacobian Dependence of Power Series

by

A. PŁOSKI

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Summary. Let $D = a_1(x_1, x_2) \frac{\partial}{\partial x_1} + a_2(x_1, x_2) \frac{\partial}{\partial x_2}$ be a non-zero derivation of the ring of the convergent power series in the variables $x = (x_1, x_2)$. Then there exists a convergent power series $t(x)$ such that every solution $u(x)$ of the equation $Du=0$ has the form $F(t(x))$. Here $F(t)$ is a convergent power series in one variable t . The corresponding result for certain types of systems of differential equations is also obtained.

1. Notations, terminology, the main result. Let K be a non-discrete valued field of characteristic zero. We will consider convergent power series f, g, \dots with coefficients in K . We will write $f|g$ if f divides g . The order of f will be denoted by $\text{ord } f$. For any convergent power series f_1, \dots, f_m we will denote by $\frac{J(f_1, \dots, f_m)}{J(x_1, \dots, x_n)}$ the Jacobian matrix of f_1, \dots, f_m with respect to the variables x_1, \dots, x_n .

Note that $\text{rank } \frac{J(f_1, \dots, f_m)}{J(x_1, \dots, x_n)} = \max r$ such that

$\det \frac{J(f_{i_1}, \dots, f_{i_r})}{J(x_{j_1}, \dots, x_{j_r})} \neq 0$ for some $1 \leq i_1 < i_2 < \dots < i_r \leq m$, $1 \leq j_1 < j_2 < \dots < j_r \leq n$.

Where it is not misleading we shall write $J(f_1, \dots, f_m)$ instead of $\frac{J(f_1, \dots, f_m)}{J(x_1, \dots, x_n)}$.

The power series f_1, \dots, f_m will be called J -dependent if $\text{rank } \frac{J(f_1, \dots, f_m)}{J(x_1, \dots, x_n)} < m$.

Otherwise they will be called J -independent. Let $D_i = \sum_{j=1}^n a_{ij}(x_1, \dots, x_n) \frac{\partial}{\partial x_j}$, $i=1, \dots, m$ be derivations of the ring of the convergent power series $K\{x_1, \dots, x_n\}$. Here $a_{ij}(x_1, \dots, x_n)$ are convergent power series, m, n are arbitrary non-negative integers.

Set

$$V = \{u(x) \in K\{x_1, \dots, x_n\} : D_i u(x) = 0 \text{ for } i=1, \dots, m\}.$$

Let r be the largest non-negative integer such that there exist J -independent power series $v_1(x), \dots, v_r(x) \in V$. Obviously $0 \leq r \leq n$. The aim of this paper is to prove the following.

THEOREM (1.1). *Let us assume that $r \leq 1$. Then there exists a power series $t(x) \in V$ without a constant term such that for every $u(x) \in V$ there exists a convergent power series $F(t)$ in one variable t such that $u(x) = F(t(x))$.*

Let $v_1(x), \dots, v_r(x) \in V$ be J -independent power series. A straightforward calculation shows that

$$V = \{u(x) \in K\{x_1, \dots, x_n\} : \text{rank } J(v_1, \dots, v_r, u) \leq r\}.$$

Hence (1.1) results from the following

THEOREM (1.2). *Let $t(x_1, \dots, x_n) \neq 0$ be a convergent power series without a constant term. Set $V_{t(x)} = \{f(x) \in K\{x\} : \text{rank } J(t, f) \leq 1\}$. The following three conditions are equivalent:*

- (1.2.1) *For every $g(x) \in V_{t(x)}$ $\text{ord } g(x) \geq \text{ord } t(x)$,*
- (1.2.2) *For every $g(x) \in V_{t(x)}$ $t(x) | g(x)$,*
- (1.2.3) *For every $g(x) \in V_{t(x)}$ there exist a convergent power series $F(t)$ in one variable t such that $g(x) = F(t(x))$.*

To prove that (1.2) implies (1.1) take $t(x) \in V$ such that $\text{ord } t(x) \geq 1$ is minimal. Then clearly $V = V_{t(x)}$.

The proof of (1.2) is given in a further section. We conclude this section with

COROLLARY (1.3). *Let $f_i(x_1, \dots, x_n)$, $i=1, \dots, m$ be a convergent power series. Suppose that*

$$\text{rank } \frac{J(f_1, \dots, f_m)}{J(x_1, \dots, x_n)} \leq 1.$$

Then there exist convergent power series $F_i(t)$, $i=1, \dots, m$ in one variable t and a convergent series $t(x)$ without a constant term such that

$$f_i(x) = F_i(t(x)) \quad \text{for } i=1, \dots, m.$$

We may assume without any loss of generality that $\frac{\partial f_1}{\partial x_1} \neq 0$.

To prove (1.3) it is sufficient to apply (1.1) to the derivations

$$D_i = \frac{\partial f_1}{\partial x_n} \frac{\partial}{\partial x_i} - \frac{\partial f_1}{\partial x_i} \frac{\partial}{\partial x_n} \quad i=1, \dots, n-1.$$

Remark. The preceding considerations give rise to the following problem. Do the power series $t_1(x), \dots, t_r(x) \in V$ without a constant term exist such that $V = K\{t_1(x), \dots, t_r(x)\}$?

Theorem (1.1) states that the answer to the problem is positive for $r \leq 1$.

2. Proof of Theorem (1.2). We shall start with the following:

LEMMA (2.1). Let $f(x), g(x)$ be convergent power series in the variables $x = (x_1, \dots, x_n)$ such that $f(0) = g(0) = 0$. Then the following three conditions are equivalent:

$$(2.1.1) \text{ rank } J(f, g) \leq 1,$$

(2.1.2) There exists a convergent power series $P(u, v) \neq 0$ such that $P(f(x), g(x)) = 0$,

(2.1.3) There exist a formal power series $\tilde{P}(u, v) \neq 0$ such that $\tilde{P}(f(x), g(x)) = 0$.

It is obvious that (2.1.2) \Rightarrow (2.1.3) and (2.1.3) \Rightarrow (2.1.1).

Proof of (2.1.1) \Rightarrow (2.1.2). We may assume, after a linear change of variables x , that $f(0, \dots, 0, x_n) \neq 0$. Then, according to one of the forms of the preparation theorem (Narasimhan [3], p. 10, Theorem 1) the ring $K\{x\}$ is a finite $K(x', f(x))$ -module. Here $x' = (x_1, \dots, x_{n-1})$. Furthermore, there exists a monic polynomial $P(x', u; v) \in K\{x', u\}[v]$ such that $P(x', f(x); g(x)) = 0$. If we choose the polynomial $P(x', u; v)$ to have a minimal non-zero degree we claim that $\frac{\partial P}{\partial x_i}(x', u; v) = 0$ for $i = 1, \dots, n-1$. In fact, it is easy to check by a differentiation of the equality $P(x', f(x); g(x)) = 0$ in variables x_1, \dots, x_{n-1} that $\frac{\partial P}{\partial x_i}(x', f(x); g(x)) = 0$ for $i = 1, \dots, n-1$. Since $P(x', u; v)$ is a monic polynomial (Zariski—Samuel [4], p. 280, Theorem 4) the degree of $\frac{\partial P}{\partial x_i}(x', u; v)$ is less than the degree of $P(x', u; v)$, hence by our choice of $P(x', u; v)$ we have $\frac{\partial P}{\partial x_i}(x', u; v) = 0$ for $i = 1, \dots, n-1$. This completes the proof of the lemma.

Remark. The equivalence between (2.1.2) and (2.1.3) was established by Abhyankar and Van der Put [1].

LEMMA (2.2). Let $f(x) \neq 0, g(x) \neq 0$ be convergent power series without constant terms. Assume that there exists a convergent power series $P(u, v) \neq 0$ such that $P(f(x), g(x)) = 0$.

Then

$$(*) \quad f(x) | g(x) \quad \text{or} \quad g(x) | f(x).$$

Since condition (*) is invariant under a K -linear nonsingular transformation of $f(x), g(x)$ then we may assume, by the Weierstrass preparation theorem, that

$$P(u, v) = v^d + A_1(u)v^{d-1} + \dots + A_d(u), \quad d = \text{ord } P(u, v).$$

From the above assumption it follows that $\text{ord } A_j(u) \geq j$ for $j = 1, \dots, d$. Consequently, we may write

$$g(x)^d + B_1(f(x))f(x)g(x)^{d-1} + \dots + B_d(f(x))f(x)^d = 0,$$

where $B_j(u)$ $j = 1, \dots, d$ are convergent power series. This implies that $f(x) | g(x)$ since the ring of power series is integrally closed.

To complete the proof of (1.2) notice that (1.2.1) \Rightarrow (1.2.2) follows from (2.1) and (2.2). Since (1.2.2) \Rightarrow (1.2.1) is obvious it suffices to prove (1.2.2) \Rightarrow (1.2.3).

In order to prove this we construct by induction a sequence $c_j \in K$ $j = 1, 2, \dots$ such that

$$f(x) \equiv \sum_{j=1}^k c_j t(x)^j \pmod{(t(x)m\{x\})} \quad \text{for } k=1, 2, \dots$$

Hence $m\{x\}$ denotes the maximal ideal of the ring $K\{x\}$. Suppose that there exist $c_1, \dots, c_k \in K$ and a power series $q(x)$ without a constant term such that

$$f(x) = \sum_{j=1}^k c_j t(x)^j + q(x) t(x).$$

It is easy to see that $q(x) \in V_{t(x)}$. In virtue of (1.2.2) there exists a power series $q'(x)$ such that $q(x) = q'(x)t(x)$; it is sufficient to set $c_{k+1} = q'(0)$.

Set $F(t) = \sum_{k=1}^{+\infty} c_k t^k$. Then clearly $f(x) = F(t(x))$.

The series $F(t)$ is convergent. In fact, from (2.1) and the Weierstrass preparation theorem it follows that the series $F(t)$ is integral algebraic over the ring $K\{t\}$, whence it is convergent (Nagata [3]).

INSTITUTE OF MATHEMATICS, M. CURIE-SKŁODOWSKA UNIVERSITY, NOWOTKI 10, 20-031 LUBLIN
(INSTYTUT MATEMATYKI, UNIWERSYTET M. CURIE-SKŁODOWSKIEJ, LUBLIN)

REFERENCES

- [1] M. Artin, *Algebraic spaces* [mimeographed], Yale University, 1969.
- [2] M. Nagata, *Some remarks on local rings, II*, Mem. Coll. Sci., Univ. Kyoto, 28 (1953), 109–120.
- [3] R. Narasimhan, *Introduction to the theory of analytic spaces*, Springer-Verlag, 1966.
- [4] O. Zariski, P. Samuel, *Commutative algebra*, vol. II, Van Nostrand, Princeton, New Jersey, 1959–1961.

А. Плоски, О якобианской зависимости степенных рядов

Содержание. Пусть $D = a_1(x_1, x_2) \frac{\partial}{\partial x_1} + a_2(x_1, x_2) \frac{\partial}{\partial x_2}$ – неупадочное дифференциальное в кольце сходящихся степенных рядов от переменных $x = (x_1, x_2)$. Тогда существует сходящийся ряд $t(x)$ такой, что каждое решение $u(x)$ уравнения $Du = 0$ имеет вид $F(t(x))$ (где $F(t)$ сходящийся ряд от одной переменной). Указаны условия при которых соответствующий результат справедлив для систем дифференциальных уравнений.