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ON THE POLAR QUOTIENTS OF AN ANALYTIC PLANE CURVE

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Abstract

We give explicit formulae in terms of characteristics and intersection multiplicities of branches for the polar quotients of a plane many–branched singularity.

Introduction

We use standard notations: $\boldsymbol{C}\{X, Y\}$ is the ring of convergent power series, $\operatorname{ord} f$ is the order of $f = f(X, Y) \in \boldsymbol{C}\{X, Y\}$, $(f, \phi)_0$ denotes the intersection multiplicity of f with ϕ. Recall that $(f, \phi)_0 \geq (\operatorname{ord} f)(\operatorname{ord} \phi)$ with equality if and only if f, ϕ are transverse. We put $\operatorname{ord} 0 = +\infty$ and use usual conventions on the symbol $+\infty$.

Let $f = f(X, Y) \in \boldsymbol{C}\{X, Y\}$ be a reduced power series (i.e. without multiple factors) and let $t = t(X, Y) \in \boldsymbol{C}\{X, Y\}$ be a regular parameter (i.e. a series of order 1) such that t does not divide f. We consider the set of polar quotients of f with respect to t:

$$Q(f, t) = \left\{ \frac{(f, \phi)_0}{(t, \phi)_0} : \phi \text{ is an irreducible factor of } \frac{\partial(t, f)}{\partial(X, Y)} \right\}.$$

The polar quotients were studied by many authors ([T], [M], [E], [Eph], [Ca], [LMW1], [LMW2], [Ga], [D], [LP]). Recently, an explicit formula for the maximal polar quotient $q(f, t) = \sup Q(f, t)$ has been given ([P], Theorem 1.3). In this note we give similar formulae for all polar quotients. Like in [P] our main tool is the Kuo–Lu lemma ([KL], Lemma 3.3 and Section 2 of this paper). Instead of using Puiseux' trees (see [KL], [E]) we make our calculations by means of Puiseux' date (Section 3). The notion of symmetric power explained in [Wh], Appendix V turns out very useful.

Let A be a nonempty set. Then A^p_{sym} is the set of all p-tuples regarded as unordered. If $\alpha = \langle a_1, \dots, a_p \rangle \in A^p_{\mathrm{sym}}$ then $|\alpha| = \{a_1, \dots, a_p\}$ is the set corresponding to α. We put $\deg \alpha = p$. If $\alpha = \langle a_1, \dots, a_p \rangle$ and $\beta = \langle b_1, \dots, b_q \rangle$

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then $\alpha \oplus \beta = \langle a_1, \dots, a_p, b_1, \dots, b_q \rangle$. Clearly $|\alpha \oplus \beta| = |\alpha| \cup |\beta|$ and $\deg(\alpha \oplus \beta) = \deg \alpha + \deg \beta$. For every positive integer m we put $m\alpha = \alpha \oplus \cdots \oplus \alpha$ (m times). Instead of $m_1\langle a_1 \rangle \oplus \cdots \oplus m_p\langle a_p \rangle = \langle \underbrace{a_1, \dots, a_1}_{m_1 \text{ times}}, \dots, \underbrace{a_p, \dots, a_p}_{m_p \text{ times}} \rangle$ we write $\langle a_1 : m_1, \dots, a_p : m_p \rangle$.

1. Main result

The semigroup $\Gamma_0(f)$ of an irreducible power series $f \in \boldsymbol{C}\{X, Y\}$ is the set of all intersection numbers $(f, \phi)_0$ where ϕ runs over all power series $\phi \in \boldsymbol{C}\{X, Y\}$ such that f does not divide ϕ. Let $t = t(X, Y)$ be a regular parameter. Put $p = (t, f)_0$. Let $\bar{b}_0, \dots, \bar{b}_h$ be the p-minimal system of generators of $\Gamma_0(f)$ determined by conditions

(i) $\bar{b}_0 = p$,

(ii) $\bar{b}_k = \min(\Gamma_0(f) \setminus (\boldsymbol{N}\bar{b}_0 + \cdots + \boldsymbol{N}\bar{b}_{k-1}))$ for $k = 1, \dots, h$,

(iii) $\Gamma_0(f) = \boldsymbol{N}\bar{b}_0 + \cdots + \boldsymbol{N}\bar{b}_h$.

Let $B_k = \mathrm{GCD}(\bar{b}_0, \dots, \bar{b}_k)$ be the greatest common divisor of $\bar{b}_0, \dots, \bar{b}_k$.

Proposition 1.1 ([M], [Eph]). *Let $f \in \boldsymbol{C}\{X, Y\}$ be an irreducible power series with* $\operatorname{ord} f > 1$. *Then*

$$Q(f, t) = \left\{ \frac{B_{k-1}\bar{b}_k}{\bar{b}_0} : k = 1, \dots, h \right\}.$$

The above formula was given by Merle ([M], Théorème 3.1) in transverse case. Then Ephraim ([Eph], Lemma 1.6) observed that Merle's result holds also for polar quotients with respect to a regular parameter t. In Section 4 we give a new proof of (1.1). Recall that the sequence $B_{k-1}\bar{b}_k/\bar{b}_0$ is strictly increasing.

If $\operatorname{ord} f = 1$ then $Q(f, t) = \emptyset$ if f, t are transverse and $Q(f, t) = \{1\}$ if f, t are not.

Remark 1.2. Using the inversion formulae for plane branches (see [P], proof of Proposition 1.1) one can calculate the polar quotients of a branch $f = 0$ in terms of the minimal system of generators $\bar{\beta}_0, \dots, \bar{\beta}_g$ of $\Gamma_0(f)$. Let $e_k = \mathrm{GCD}(\bar{\beta}_0, \dots, \bar{\beta}_k)$ for $k = 0, 1, \dots, g$. Then

$$Q(f, t) = \left\{ \frac{e_{k-1}\bar{\beta}_k}{\bar{\beta}_0} : k = 1, \dots, g \right\} \cup O(f, t)$$

where $O(f, t) = \emptyset$ if $(f, t)_0 = \bar{\beta}_0$ or $(f, t)_0 = \bar{\beta}_1$ and $O(f, t) = \{\operatorname{ord} f\}$ otherwise.

Let $\phi, \psi \in \boldsymbol{C}\{X, Y\}$ be irreducible power series. The contact coefficient (in the sense of Hironaka) with respect to a regular parameter $t \in \boldsymbol{C}\{X, Y\}$ is the rational number

$$h(\phi, \psi; t) = \frac{(\phi, \psi)_0}{(t, \psi)_0}$$

In [LMW1] this notion is introduced in the case where t and ψ are transverse. Then $h(\phi,\psi;t) = (\phi,\psi)_0/\operatorname{ord}\psi$ and we write $h(\phi,\psi)$ instead of $h(\phi,\psi;t)$.

Let us consider a reduced power series $f = f_1 \cdots f_r$ with $r > 1$ irreducible factors. Fix $i \in \{1,\dots,r\}$. For every real number $\tau > 0$ we put $J_i(\tau) = \{j \in \{1,\dots,r\} : h(f_i, f_j; t) \le \tau\}$ and $J_i(\tau)^c = \{1,\dots,r\} \backslash J_i(\tau)$.

By convention $h(f_i, f_i; t) = +\infty$, hence $i \in J_i(\tau)^c$. We put

$$q_i(\tau) = \frac{1}{(t, f_i)_0}\left\{\left(\sum_{j \in J_i(\tau)^c} (t, f_j)_0\right)\tau + \sum_{j \in J_i(\tau)} (f_i, f_j)_0\right\} \quad \text{for } \tau > 0.$$

To write the above formula more explicitly suppose $i = 1$ and assume $h(f_1, f_r; t) \le h(f_1, f_{r-1}; t) \le \cdots \le h(f_1, f_2; t) < h(f_1, f_1; t) = +\infty$. Then

$$q_1(\tau) = \begin{cases} \dfrac{(t, f_1 \cdots f_j)_0}{(t, f_1)_0}\tau + \dfrac{(f_1, f_{j+1} \cdots f_r)_0}{(t, f_1)_0} & \text{if } h(f_1, f_{j+1}; t) \le \tau < h(f_1, f_j; t),\ j < r \\ \dfrac{(t, f)_0}{(t, f_1)_0}\tau & \text{if } \tau < h(f_1, f_r; t) \end{cases}$$

It is easy to see that the functions q_i are piecewise linear, continuous and strictly increasing. Note that q_i is determined by the intersection multiplicities of i-th branch $f_i = 0$ with the remaining branches and with the smooth curve $t = 0$. Let $H_i(f,t) = \{h(f_i, f_j; t) : j \ne i\}$ be the set of contact coefficients of the branch $f_i = 0$ with branches $f_j = 0$ $(j \ne i)$. The main results of this paper is

Theorem 1.3. *Let $f = f_1 \cdots f_r$ be a reduced power series with $r > 1$ irreducible factors. Then*

$$Q(f,t) = \bigcup_{i=1}^{r} q_i(Q(f_i,t) \cup H_i(f,t)).$$

We give the proof of (1.3) in Section 4. We put $Q_i(f,t) = q_i(Q(f_i,t) \cup H_i(f,t))$ and call the elements of $Q_i(f,t)$ polar quotients associated with the branch $f_i = 0$.

A polar quotient can be associated with more than one branch. From Theorem 1.3 it follows that the polar quotients of the curve $f = 0$ with respect to a regular parameter t depend only on the equisingularity type of the curve $tf = 0$.

If t and f are transverse then the polar quotients of $f = 0$ do not depend on the regular parameter t and are determined by the equisingularity type of the curve $f = 0$. Then we write $Q_i(f)$ instead of $Q_i(f,t)$ and $Q(f)$ instead of $Q(f,t)$.

Using (1.3) we obtain easily the formula for the maximal polar quotient $q(f,t) = \sup Q(f,t)$ given in [P]. Obviously $q(f,t) = \max_{i=1}^{r}\{\max Q_i(f,t)\}$, hence it suffices to calculate the maximal polar quotients associated with branches.

COROLLARY 1.4.

$$\max Q_i(f,t) = \max\left\{q(f_i,t), \max_{j\neq i}\{h(f_i,f_j;t)\}\right\} + \frac{1}{(t,f_i)_0}\sum_{j\neq i}(f_i,f_j)_0.$$

Proof. Let $\tau_i = \max(Q(f_i,t)\cup H_i(f,t)) = \max\{q(f_i,t), \max_{j\neq i}\{h(f_i,f_j;t)\}\}$. Then $J_i(\tau_i) = \{1,\dots,r\}\backslash\{i\}$ and $q_i(\tau_i) = \tau_i + (1/(t,f_i)_0)\sum_{j\neq i}(f_i,f_j)_0$. We get then $\max Q_i(f,t) = q_i(\tau_i)$ for q_i is an increasing function.

Note that for every $i\in\{1,\dots,r\}$ the rational numbers $q(f_i,t)+(1/(t,f_i)_0)\cdot\sum_{j\neq i}(f_i,f_j)_0$ (provided that $q(f_i,t)\neq-\infty$) and $\max_{j\neq i}\{h(f_i,f_j;t)\}+(1/(t,f_i)_0)\cdot\sum_{j\neq i}(f_i,f_j)_0$ are polar quotients of $f=0$.

COROLLARY 1.5. *If $f = f_1\cdots f_r$ $(r>1)$ with transverse factors f_i, f_j for $i\neq j$ then*

$$Q(f) = \bigcup_{i=1}^{r}\{\tau_i + \operatorname{ord} f - \operatorname{ord} f_i : \tau_i\in Q(f_i)\}\cup\{\operatorname{ord} f\}.$$

Proof. Let us calculate $Q_1(f)$. It is easy to see that $J_1(\tau) = \{j\in\{2,\dots,r\} : \operatorname{ord} f_1\leq\tau\}$. Hence $J_1(\tau)=\emptyset$ if $\tau<\operatorname{ord} f_1$ and $J_1(\tau)=\{2,\dots,r\}$ if $\tau\geq\operatorname{ord} f_1$. Obviously $H_1(f)=\{\operatorname{ord} f_1\}$. Therefore we get

$$q_1(\tau) = \begin{cases} \tau + \operatorname{ord} f - \operatorname{ord} f_1 & \text{if } \tau\geq\operatorname{ord} f_1 \\ \dfrac{\operatorname{ord} f}{\operatorname{ord} f_1}\tau & \text{if } \tau<\operatorname{ord} f_1 \end{cases}$$

For every $\tau_1\in Q(f_1)$ one gets $\tau_1\geq\operatorname{ord} f_1$. Consequently $Q_1(f) = \{q_1(\tau_1) : \tau_1\in Q(f_1)\}\cup q_1(\operatorname{ord} f_1) = \{\tau_1+\operatorname{ord} f-\operatorname{ord} f_1 : \tau_1\in Q(f_1)\}\cup\{\operatorname{ord} f\}$ and the corollary follows.

To give another application of the main result we calculate polar quotients of "diagonal curves" introduced by Delgado [D].

COROLLARY 1.6. *Let $f=f_1\cdots f_r$ be a reduced power series. Suppose that $\Gamma_0(f_i)$ and $(t,f_i)_0$ do not depend on $i\in\{1,\dots,r\}$, $\Gamma_0(f_i)=\langle\bar b_0,\bar b_1,\dots,\bar b_h\rangle$ and $(f_i,f_j)_0 = B_{h-1}\bar b_h + c$ with $c>0$, for $i\neq j$. Then*

$$Q(f,t) = \left\{r\frac{B_{k-1}\bar b_k}{\bar b_0} : k=1,\dots,h\right\}\cup\left\{r\frac{B_{h-1}\bar b_h + c}{\bar b_0}\right\}.$$

Proof. By Proposition 1.1 we get $Q(f_i,t)=\{B_{k-1}\bar b_k/\bar b_0 : k=1,\dots,h\}$ on the other hand $H_i(f,t)=\{(B_{h-1}\bar b_h+c)/\bar b_0\}$. It is easy to check that $q_i(\tau)=r\tau$ if $\tau<(B_{h-1}\bar b_h+c)/\bar b_0$ and $q_i(\tau)=\tau+(r-1)(B_{h-1}\bar b_h+c)/\bar b_0$ if $\tau\geq(B_{h-1}\bar b_h+c)/\bar b_0$. Now we get (1.6) from Theorem 1.3.

2. Zeros of a derivative

Let $\boldsymbol{C}\{X\}^* = \bigcup_{n\geq 1} \boldsymbol{C}\{X^{1/n}\}$ be the ring of Puiseux series. If $f = f(X,Y) \in \boldsymbol{C}\{X,Y\}$ is a power series Y-regular of order $p > 0$ i.e. such that $\operatorname{ord} f(0,Y) = p$ then the equation $f(X,Y) = 0$ has in $\boldsymbol{C}\{X\}^*$ p roots $y_1 = y_1(X), \dots, y_p = y_p(X)$ of positive order counted with multiplicities. We put $\operatorname{Zer} f = \langle y_1, \dots, y_p \rangle$. Let $S \subset \boldsymbol{C}\{X\}^*$ be a set of Puiseux series. If $y_1, \dots, y_q \in S$ and $y_{q+1}, \dots, y_p \notin S$ then we put $\operatorname{Zer} f \cap S = \langle y_1, \dots, y_q \rangle$. Let us recall

The Kuo–Lu Lemma ([KL], Lemma 3.3). *Suppose that $f = f(X,Y)$ is Y-regular of order $p > 1$. Let $\operatorname{Zer} f = \langle y_1, \dots, y_p \rangle$ and $\operatorname{Zer}(\partial f/\partial Y) = \langle z_1, \dots, z_{p-1} \rangle$. Then for every $i \in \{1, \dots, p\}$:*

$$\langle \operatorname{ord}(z_1 - y_i), \dots, \operatorname{ord}(z_{p-1} - y_i) \rangle$$
$$= \langle \operatorname{ord}(y_1 - y_i), \dots, \widehat{\operatorname{ord}(y_i - y_i)}, \dots, \operatorname{ord}(y_p - y_i) \rangle.$$

It is convenient to prove the lemma for formal power series with coefficients in $\boldsymbol{C}\{X\}^*$. Replacing $f(X,Y)$ by $f(X, Y + y_i)$ we may assume $y_i = 0$. Then the series $f(X,Y)/Y$ and $\partial f/\partial Y$ have the same Newton diagram (see [Wa], Chapter IV) and the lemma follows from the fact that the Newton diagram determines the orders of roots and the number of roots of given order. For more details see [GP1].

For every $u \in \boldsymbol{C}\{X\}^*$ and $r > 0$ we put

$$S(u,r) = \{v \in \boldsymbol{C}\{X\}^* : \operatorname{ord}(u - v) = r\},$$
$$B(u,r) = \{v \in \boldsymbol{C}\{X\}^* : \operatorname{ord}(u - v) > r\},$$
$$\bar{B}(u,r) = \{v \in \boldsymbol{C}\{X\}^* : \operatorname{ord}(u - v) \geq r\}.$$

According to the given definitions $\deg(\operatorname{Zer} f \cap S)$ is equal to the number of roots counted with multiplicities of the equation $f(X,Y) = 0$ lying in the set S of Puiseux series. If $|\operatorname{Zer} f| \cap S = \emptyset$ then we put $\deg(\operatorname{Zer} f \cap S) = 0$.

Lemma 2.1. *Suppose that $f = f(X,Y)$ is Y-regular of order $p > 1$. Then for every $y \in \operatorname{Zer} f$ and for every $r > 0$*

(i) $\deg(\operatorname{Zer} f \cap S(y,r)) = \deg(\operatorname{Zer}(\partial f/\partial Y) \cap S(y,r))$,

(ii) $\deg(\operatorname{Zer} f \cap B(y,r)) = \deg(\operatorname{Zer}(\partial f/\partial Y) \cap B(y,r)) + 1$,

(iii) $\deg(\operatorname{Zer} f \cap \bar{B}(y,r)) = \deg(\operatorname{Zer}(\partial f/\partial Y) \cap \bar{B}(y,r)) + 1$.

Proof. The part (i) is a reformulation of the Kuo–Lu lemma, (ii) and (iii) follow from (i).

Proposition 2.2. *Under assumptions of Lemma* 2.1:

(i) *for every $z \in |\operatorname{Zer}(\partial f/\partial Y)|$ there exist $y_1, y_2 \in \operatorname{Zer} f$, $y_1 \neq y_2$ such that $\operatorname{ord}(z - y_1) = \operatorname{ord}(y_1 - y_2) \geq \operatorname{ord}(z - y)$ for all $y \in |\operatorname{Zer} f|$,*

(ii) *for every* $y_1, y_2 \in \mathrm{Zer}\, f$, $y_1 \neq y_2$ *there exists* $z \in |\mathrm{Zer}(\partial f/\partial Y)|$ *such that* $\mathrm{ord}(z - y_1) = \mathrm{ord}(y_1 - y_2) \geq \mathrm{ord}(z - y)$ *for all* $y \in |\mathrm{Zer}\, f|$.

Proof. (i) Fix $z \in |\mathrm{Zer}(\partial f/\partial Y)|$. Let $y_1 \in |\mathrm{Zer}\, f|$ be such that $\mathrm{ord}(z - y_1) = \max\{\mathrm{ord}(z - y) : y \in |\mathrm{Zer}\, f|\}$. By the Kuo–Lu lemma there exists $y_2 \in |\mathrm{Zer}\, f|$ such that $\mathrm{ord}(z - y_1) = \mathrm{ord}(y_1 - y_2)$ and we are done.

(ii) Suppose $y_1 \neq y_2$ are given roots of the equation $f(X, Y) = 0$.

Let $\bar{B} = \bar{B}(y_1, \mathrm{ord}(y_1 - y_2))$. Let $B_1, \ldots, B_s$ be a sequence of pairwise disjoint balls such that $\bigcup_{i=1}^{s} B_i = \bigcup\{B(y, \mathrm{ord}(y_1 - y_2)) : y \in |\mathrm{Zer}\, f| \cap \bar{B}\}$.

Note that

1) $s \geq 2$ for $B(y_1, \mathrm{ord}(y_1 - y_2)) \cap B(y_2, \mathrm{ord}(y_1 - y_2)) = \emptyset$,
2) $B_i \subset \bar{B}$ $(i = 1, \ldots, s)$ because the center of B_i belongs to $\bar{B}$ and B_i, $\bar{B}$ have the same radius.
3) $\bigcup_{i=1}^{s}(B_i \cap |\mathrm{Zer}\, f|) = \bar{B} \cap |\mathrm{Zer}\, f|$ for B_i cover $\bar{B} \cap |\mathrm{Zer}\, f|$ and are pairwise disjoint.

Using Lemma 2.1 we calculate $\deg((\bar{B} \setminus \bigcup_{i=1}^{s} B_i) \cap \mathrm{Zer}(\partial f/\partial Y)) = \deg(\bar{B} \cap \mathrm{Zer}(\partial f/\partial Y)) - \sum_{i=1}^{s} \deg(B_i \cap \mathrm{Zer}(\partial f/\partial Y)) = (\deg(\bar{B} \cap \mathrm{Zer}\, f) - 1) - \sum_{i=1}^{s}(\deg(B_i \cap \mathrm{Zer}\, f) - 1) = \deg(\bar{B} \cap \mathrm{Zer}\, f) - \sum_{i=1}^{s} \deg(B_i \cap \mathrm{Zer}\, f) + (s - 1) = s - 1 > 0$.

Therefore the set $(\bar{B} \setminus \bigcup_{i=1}^{s} B_i) \cap |\mathrm{Zer}(\partial f/\partial Y)|$ is nonempty. Let $z \in (\bar{B} \setminus \bigcup_{i=1}^{s} B_i) \cap |\mathrm{Zer}(\partial f/\partial Y)|$. Then $\mathrm{ord}(z - y_1) \geq \mathrm{ord}(y_1 - y_2) \geq \mathrm{ord}(z - y)$ for all $y \in |\mathrm{Zer}\, f| \cap \bar{B}$. In particular $\mathrm{ord}(z - y_1) = \mathrm{ord}(y_1 - y_2)$. If $y \in |\mathrm{Zer}\, f| \setminus \bar{B}$ then $\mathrm{ord}(y - y_1) < \mathrm{ord}(y_1 - y_2) = \mathrm{ord}(z - y_1)$. Hence we get $\mathrm{ord}(z - y_1) \geq \mathrm{ord}(z - y)$ for all $y \in |\mathrm{Zer}\, f|$. Thus (ii) follows.

For given $y_1, y_2 \in |\mathrm{Zer}\, f|$, $y_1 \neq y_2$ we put

$$L_{y_1,y_2} = |\mathrm{Zer}\, f| \cap \bar{B}(y_1, \mathrm{ord}(y_1 - y_2)),$$
$$L^c_{y_1,y_2} = |\mathrm{Zer}\, f| \setminus L_{y_1,y_2},$$
$$l_{y_1,y_2} = \sum_{y \in L^c_{y_1,y_2}} \mathrm{ord}(y_1 - y) + (\sharp L_{y_1,y_2})\, \mathrm{ord}(y_1 - y_2).$$

THEOREM 2.3. *Let* $f = f(X, Y)$ *be a reduced* Y*-regular power series with* $\mathrm{ord}\, f(0, Y) > 1$. *Then* $\{l_{y_1,y_2} : y_1, y_2 \in |\mathrm{Zer}\, f|, y_1 \neq y_2\} = \{\mathrm{ord}\, f(X, z(X)) : z \in |\mathrm{Zer}(\partial f/\partial Y)|\}$.

Proof. For every $z = z(X) \in \boldsymbol{C}\{X\}^*$ without constant term one has

$$\mathrm{ord}\, f(X, z(X)) = \sum_{y \in |\mathrm{Zer}\, f|} \mathrm{ord}(z - y) = \sum_{y \in L^c_{y_1,y_2}} \mathrm{ord}(z - y) + \sum_{y \in L_{y_1,y_2}} \mathrm{ord}(z - y). \tag{1}$$

Let $y_1, y_2 \in \mathrm{Zer}\, f$ and $z \in |\mathrm{Zer}(\partial f/\partial Y)|$ be such that $\mathrm{ord}(z - y_1) = \mathrm{ord}(z - y_2) \geq \mathrm{ord}(z - y)$ for all $y \in |\mathrm{Zer}\, f|$. We will check that

$$\mathrm{ord}(z - y) = \begin{cases} \mathrm{ord}(y - y_1) & \text{if } y \in L^c_{y_1,y_2} \\ \mathrm{ord}(y_2 - y_1) & \text{if } y \in L_{y_1,y_2} \end{cases} \tag{2}$$

Indeed, if $y \in L^c_{y_1,y_2}$ then $\mathrm{ord}(z-y) = \mathrm{ord}(y-y_1)$ for $\mathrm{ord}(y-y_1) < \mathrm{ord}(y_2-y_1) = \mathrm{ord}(z-y_1)$.

If $y \in L_{y_1,y_2}$ then

$$\begin{aligned}\mathrm{ord}(z-y) \geq \min\{\mathrm{ord}(z-y_1), \mathrm{ord}(y_1-y)\} &= \min\{\mathrm{ord}(y_2-y_1), \mathrm{ord}(y_1-y)\} \\ &= \mathrm{ord}(y_2-y_1).\end{aligned}$$

On the other hand $\mathrm{ord}(z-y) \leq \mathrm{ord}(z-y_1)$ for all $y \in |\mathrm{Zer}\, f|$ and $\mathrm{ord}(z-y) \leq \mathrm{ord}(y_2-y_1)$. Hence $\mathrm{ord}(z-y) = \mathrm{ord}(y_2-y_1)$ for $y \in L_{y_1,y_2}$. Now, Theorem 2.3 follows from (1), (2) and Proposition 2.2.

3. Puiseux' date

We use the notations and definitions from [P], Section 3. Let $f = f(X,Y) \in \boldsymbol{C}\{X,Y\}$ be an irreducible power series Y-regular of order $p > 0$. Let $(b_0,\dots,b_h)$ be the characteristic of f. We put $B_k = \mathrm{GCD}(b_0,\dots,b_k)$ for $k = 0,1,\dots,h$ and $\bar{b}_k = b_k + (1/B_{k-1})\sum_{i=1}^{k-1}(B_{i-1}-B_i)b_i$ for $k = 0,1,\dots,h$. Then the sequence $\bar{b}_0 = p$, $\bar{b}_1,\dots,\bar{b}_h$ is the p-minimal system of generators of the semigroup $\Gamma_0(f)$ (see [P], Proposition 3.2). We put $P(f) = \langle b_1/b_0 : B_0 - B_1, \dots, b_h/b_0 : B_{h-1} - B_h, +\infty : 1\rangle$ and call $P(f)$ Puiseux' date of the branch $f = 0$. Clearly $|P(f)| = \{b_1/b_0,\dots,b_h/b_0,+\infty\}$. If $p = 1$ then $P(f) = \langle +\infty\rangle$.

Proposition 3.1. *Let* $\mathrm{Zer}\, f = \langle y_1,\dots,y_p\rangle$. *Then for every* $i \in \{1,\dots,p\}$:

$$\langle \mathrm{ord}(y_1-y_i),\dots,\mathrm{ord}(y_p-y_i)\rangle = P(f).$$

Proof. It is a reformulation of Proposition 3.1 from [P].

The following notation will be useful. Let $\alpha \in \boldsymbol{R}^p_{\mathrm{sym}}$ and let $K > 0$ be a real number. Write $\alpha = \langle a_1,\dots,a_q,a_{q+1},\dots,a_p\rangle$ where $a_1,\dots,a_q < K$ and $K \leq a_{q+1},\dots,a_p$. Then we put $\alpha_K = \langle a_1,\dots,a_q,K,\dots,K\rangle \in \boldsymbol{R}^p_{\mathrm{sym}}$ with K repeated $p-q$ times. We let $\alpha_{(+\infty)} = \alpha$. Clearly $(\alpha \oplus \beta)_K = \alpha_K \oplus \beta_K$ and $(\alpha_K)_{K'} = \alpha_{\min(K,K')}$.

Let $g = g(X,Y) \in \boldsymbol{C}\{X,Y\}$ be an irreducible Y-regular power series. Let $\mathrm{Zer}\, g = \langle z_1(X),\dots,z_q(X)\rangle$. Then the order of contact of f and g is defined to be

$$\mathrm{cont}(f,g) = \max\{\mathrm{ord}(y_i(X) - z_j(X)) : 1 \leq i \leq p, 1 \leq j \leq q\}.$$

Note that for every $K > 0$

$$P(f)_K = \left\langle \frac{b_1}{b_0} : B_0 - B_1, \dots, \frac{b_{k-1}}{b_0} : B_{k-2} - B_{k-1}, K : B_{k-1}\right\rangle$$

where $k > 0$ is the smallest integer such that $K \leq b_k/b_0$. We put $b_{h+1}/b_0 = +\infty$.

Proposition 3.2. *Let* $z = z(X) \in |\mathrm{Zer}\, g|$. *Then*

$$\langle \mathrm{ord}(z-y_1),\dots,\mathrm{ord}(z-y_p)\rangle = P(f)_{\mathrm{cont}(f,g)}.$$

Proof. It is easy to check that $\max_{i=1}^{p}\{\mathrm{ord}(z-y_i)\} = \mathrm{cont}(f,g)$. We may assume that $\mathrm{cont}(f,g) = \mathrm{ord}(z-y_p)$. Then

$$\langle \mathrm{ord}(z-y_1), \ldots, \mathrm{ord}(z-y_p)\rangle = \langle \mathrm{ord}(y_1-y_p), \ldots, \mathrm{ord}(y_p-y_p)\rangle_{\mathrm{ord}(z-y_p)}$$

and (3.2) follows from (3.1).

For every $\alpha = \langle a_1, \ldots, a_p\rangle \in \boldsymbol{R}^p_{\mathrm{sym}}$ we put $\sum \alpha = \sum_{i=1}^{p} a_i$. Clearly $\sum(\alpha \oplus \beta) = \sum \alpha + \sum \beta$. Note that $K \mapsto \sum \alpha_K$ is strictly increasing for $K > 0$.

PROPOSITION 3.3. *Let $f = f(X,Y) \in \boldsymbol{C}\{X,Y\}$ and $g = g(X,Y) \in \boldsymbol{C}\{X,Y\}$ be irreducible Y-regular power series. Suppose f is of characteristic $(b_0, \ldots, b_h)$. Then*

(i) $\sum P(f)_{b_k/b_0} = B_{k-1}\bar{b}_k/\bar{b}_0$,
(ii) $\sum P(f)_{\mathrm{cont}(f,g)} = (f,g)_0/(X,g)_0$,
(iii) $(X,f)_0 P(g)_{\mathrm{cont}(f,g)} = (X,g)_0 P(f)_{\mathrm{cont}(f,g)}$,
(iv) *if* $K \le \mathrm{cont}(f,g)$ *then* $\sum P(g)_K = ((X,g)_0/(X,f)_0) \sum P(f)_K$.

Proof. Property (i) follows from the definitions. To check (ii) recall that $(f,g)_0/(X,g)_0 = \mathrm{ord}\, f(X, z(X))$ where $z = z(X) \in |\mathrm{Zer}\, g|$.

Hence $(f,g)_0/(X,g)_0 = \sum_{i=1}^{p} \mathrm{ord}(z(X) - y_i(X)) = \sum P(f)_{\mathrm{cont}(f,g)}$ by Proposition 3.2.

Let $k > 0$ be the smallest integer such that $\mathrm{cont}(f,g) \le b_k/b_0$ and let us consider the characteristic $(b'_0, \ldots, b'_{h'})$ of g. Then $k \le h'$, $\mathrm{cont}(f,g) \le b'_k/b'_0$ and $b'_i/b'_0 = b_i/b_0$ for $i < k$. Consequently $b'_0 B_i = b_0 B'_i$ for $i \le k$ and (iii) follows. Property (iv) is an easy consequence of (iii).

Remark 3.4. Properties (i), (ii) and (iii) listed in Proposition 3.3 are equivalent to the classical formula for the intersection multiplicity of two branches (see [P]).

Consider a reduced power series $f = f_1 \cdots f_r$ with $r > 1$ factors. For every $i \in \{1, \ldots, r\}$ we put $K_{ij} = \mathrm{cont}(f_i, f_j)$ if $i \ne j$ and $K_{ii} = +\infty$. We define Puiseux' date $P_i(f)$ of f with respect to the irreducible factor f_i by putting

$$P_i(f) = P(f_1)_{K_{i1}} \oplus \cdots \oplus P(f_r)_{K_{ir}}.$$

Proposition 3.5 justifies our definition of Puiseux' date.

PROPOSITION 3.5. *With the notation introduced above*
(i) $|P_i(f)| = |P(f_i)| \cup \{K_{i1}, \ldots, K_{ir}\}$,
(ii) *Let* $y_i \in |\mathrm{Zer}\, f_j|$ *for some* $j \in \{1, \ldots, r\}$. *Then*

$$\langle \mathrm{ord}(y_i - y_1), \ldots, \mathrm{ord}(y_i - y_i), \ldots, \mathrm{ord}(y_i - y_p)\rangle = P_j(f).$$

Proof. We have $|P_i(f)| = \bigcup_{j=1}^{r} |P(f_j)_{K_{ij}}| = \bigcup_{j=1}^{r} |P(f_i)_{K_{ij}}|$ by Proposition 3.3(iii). Let $b_0, \ldots, b_h$ be the characteristic of the branch f_i. Then $|P(f_i)_{K_{ii}}| =$

$|P(f_i)| = \{b_1/b_0, \dots, b_h/b_0, +\infty\}$ and $|P(f_i)_{K_{ij}}| = \{b_1/b_0, \dots, b_{k(j)-1}/b_0, K_{ij}\}$ where $k(j)$ is the smallest integer $k > 0$ such that $K_{ij} \le b_k/b_0$. Summing up we get $|P(f_i)| = \{b_1/b_0, \dots, b_h/b_0, K_{i1}, \dots, K_{ir}\}$ which proves (i). Part (ii) follows from (3.1), (3.2) and from the definition of $P_j(f)$.

4. Proof

A local isomorphism Φ is a pair of power series without constant term such that $\operatorname{Jac}\Phi(0,0) \neq 0$. It is easy to check the following

Lemma 4.1. *Let Φ be a local isomorphism. A rational number q is a polar quotient of f with respect to a regular parameter t if and only if q is a polar quotient of $f \circ \Phi$ with respect to $t \circ \Phi$.*

Therefore to prove (1.1) and (1.3) it suffices to consider the case $t = X$.

Proposition 4.2. *Let $f = f_1 \cdots f_r$ be a reduced power series with irreducible factors f_i. Suppose that f is Y-regular of order $p > 0$ and let $p_i : \boldsymbol{R}_+ \to \boldsymbol{R}$ be given by the formula*

$$p_i(K) = \sum P_i(f)_K \quad \text{for } K > 0.$$

Let $p_i(|P_i(f)|) = \{p_i(K) : K \in |P_i(f)| \cap \boldsymbol{R}\}$. Then $Q(f, X) = \bigcup_{i=1}^r p_i(|P_i(f)|)$.

Proof. Let $\operatorname{Zer} f = \langle y_1, \dots, y_p \rangle$ and $\operatorname{Zer}(\partial f/\partial Y) = \langle z_1, \dots, z_{p-1} \rangle$. It is easy to see that $Q(f, X) = \{\operatorname{ord} f(X, z_1(X)), \dots, \operatorname{ord} f(X, z_{p-1}(X))\}$. Indeed, if ϕ is an irreducible factor of $\partial f/\partial Y$ and $z_i = z_i(X) \in \operatorname{Zer}\phi$ then $(f, \phi)_0/(X, \phi)_0 = \operatorname{ord} f(X, z_i(X))$.

Let l_{y_i, y_j} $(i \neq j)$ be the quantities introduced in Section 3 of this paper. It is easy to see that

$$l_{y_i, y_j} = \sum \langle \operatorname{ord}(y_i - y_1), \dots, \operatorname{ord}(y_i - y_p) \rangle_{\operatorname{ord}(y_i - y_j)}$$

for $i \neq j$.

Now suppose that $y_i \in |\operatorname{Zer} f_1|$ and let $K_1 = \operatorname{ord}(y_i - y_j)$ for a $j \neq i$. By Proposition 3.5 we get

$$l_{y_i, y_j} = \sum P_1(f)_{K_1} = p_1(K_1)$$

and consequently $\{l_{y_i, y_j} : j \neq i\} = \{p_1(K_1) : K_1 \in |P_1(f)|\} = p_1(|P_1(f)|)$.

Similarly $\{l_{y_i, y_j} : j \neq i\} = p_k(|P_k(f)|)$ if $y_i \in |\operatorname{Zer} f_k|$ for $k = 2, \dots, r$. Therefore by Theorem 2.3 we get $Q(f, X) = \{l_{y_i, y_j} : j \neq i\} = p_1(|P_1(f)|) \cup \cdots \cup p_r(|P_r(f)|)$.

Now we can give

Proof of Proposition 1.1. We assume $t = X$. Let f be an irreducible power series Y-regular of order $p > 1$. Let $(b_0, b_1, \dots, b_h)$ be the characteristic of f.

According to Proposition 4.2 we have $Q(f,X)=p(|P(f)|)$ where $p:\boldsymbol{R}_+\to\boldsymbol{R}$ is given by the formula $p(K)=\sum P(f)_K$ for $K>0$. Recall that $|P(f)|=\{b_1/b_0,\dots,b_h/b_0,+\infty\}$ and $p(b_k/b_0)=\sum P(f)_{b_k/b_0}=B_{k-1}\bar{b}_k/\bar{b}_0$ by Proposition 3.3(i). Therefore $Q(f,X)=\{B_{k-1}\bar{b}_k/\bar{b}_0:k=1,\dots,h\}$.

Proof of Theorem 1.3. We assume $t=X$. We need to calculate $p_i(|P_i(f)|)$ for $i=1,\dots,r$. Without restriction of generality we assume $i=1$. Let $K>0$. By definition of Puiseux' date we get

$$P_1(f)_K=P(f_1)_{\min(K,K_{11})}\oplus\cdots\oplus P(f_r)_{\min(K,K_{1r})}$$

Therefore

$$p_1(K)=\sum P_1(f)_K=\sum_{j=1}^{r}\Bigl(\sum P(f_j)_{\min(K,K_{1j})}\Bigr)$$
$$=\sum_{j=1}^{r}\frac{(X,f_j)_0}{(X,f_1)_0}\Bigl(\sum P(f_1)_{\min(K,K_{1j})}\Bigr)$$

by Proposition 3.3(iv).

Recall that $K_{1j}\le K$ if and only if $\sum P(f_1)_{K_{1j}}\le\sum P(f_1)_K$. On the other hand by Proposition 3.3(ii) we get

$$\sum P(f_1)_{K_{1j}}=\frac{(f_1,f_j)_0}{(X,f_j)_0}.$$

Let $p_{11}(K)=\sum P(f_1)_K$. Thus $K_{1j}\le K$ if and only if $(f_1,f_j)_0/(X,f_j)_0\le p_{11}(K)$. By definition of the set $J_1(\tau)$ we see that $K_{1j}\le K$ if and only if $j\in J_1(p_{11}(K))$. Then, we can rewrite the formula for $p_1(K)$ as follows

$$p_1(K)=\sum_{j=1}^{r}\frac{(X,f_j)_0}{(X,f_1)_0}\Bigl(\sum P(f_1)_{\min(K,K_{1j})}\Bigr)$$
$$=\sum_{j\in J_1^c(p_{11}(K))}\frac{(X,f_j)_0}{(X,f_1)_0}\Bigl(\sum P(f_1)_{\min(K,K_{1j})}\Bigr)+\sum_{j\in J_1(p_{11}(K))}\Bigl(\sum P(f_1)_{\min(K,K_{1j})}\Bigr)$$
$$=\sum_{j\in J_1^c(p_{11}(K))}\frac{(X,f_j)_0}{(X,f_1)_0}p_{11}(K)+\sum_{j\in J_1(p_{11}(K))}\frac{(X,f_j)_0}{(X,f_1)_0}\frac{(f_1,f_j)_0}{(X,f_j)_0}$$
$$=\Biggl(\sum_{j\in J_1^c(p_{11}(K))}\frac{(X,f_j)_0}{(X,f_1)_0}\Biggr)p_{11}(K)+\sum_{j\in J_1(p_{11}(K))}\frac{(f_1,f_j)_0}{(X,f_1)_0}=q_1(p_{11}(K)).$$

Recall that $|P_1(f)|=|P(f_1)|\cup\{K_{12},\dots,K_{1r}\}$. Thus $p_{11}(|P_1(f)|)=p_{11}(|P_1(f)|)\cup\{p_{11}(K_{12}),\dots,p_{11}(K_{1r})\}=Q(f_1,X)\cup H_1(f,X)$ by Proposition 3.3(i), (ii) and $p_1(|P_1(f)|)=q_1(p_{11}(|P_1(f)|))=q_1(Q(f_1,X)\cup H_1(f,X))$. Analogously we get $p_i(|P_i(f)|) = q_i(Q(f_i, X) \cup H_i(f, X))$ for $i = 2, \dots, r$ and the theorem follows from Proposition 4.2.

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