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# EUCLIDEAN ALGORITHM AND POLYNOMIAL EQUATIONS AFTER LABATIE

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ABSTRACT. We recall Labatie's effective method of solving polynomial equations with two unknowns by using the Euclidean algorithm.

## INTRODUCTION

The French mathematician Labatie published in 1835 a booklet on a method of solving polynomial systems of equations in two unknowns (see [Fin1]). He used the polynomial division to replace the given system of equations by the collection of triangular systems. Labatie's theorem can be found in some old Algebra books: by Finck [Fin2], Serret [Se] and Netto [Ne], but as far as we know, not in any Algebra text book written in the twentieth century.

In this paper we recall Labatie's method following Serret [Se] (pp. 196-206). Then we give, in a modern setting, an improvement of Labatie's result due to Bonnet [Bo].

Let $\mathbf{K}$ be a field of arbitrary characteristic. We shall consider polynomials with coefficients in $\mathbf{K}$. If $W = W(x, y) \in \mathbf{K}[x, y]$ then we denote by $\deg_y W$ the degree of $W$ with respect to $y$. We say that a non-zero polynomial $W$ is *y-primitive* if it is a primitive polynomial in the ring $\mathbf{K}[x][y]$, that is, if 1 is the greatest common divisor of all the non-zero coefficients that are dependent on $x$. If $V$, $W \in \mathbf{K}[x, y]$ satisfy the condition $0 < \deg_y V \leq \deg_y W$ then there are polynomials $Q$ (quotient), $R$ (remainder) in $\mathbf{K}[x, y]$ and a non-zero polynomial $u = u(x) \in \mathbf{K}[x]$ such that $uW = QV + R$, where $\deg_y R < \deg_y V$ or $R = 0$

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The greatest common divisor of polynomials $V$, $W$ may be computed using the Euclidean algorithm, see [Bô] chapter XVI. Recently Hilmar and Smyth [H-S] gave a very simple proof of Bézout's theorem for plane projective curves using as a main tool the Euclidean division.

## 1. Euclidean algorithm

Let $V_1,\ V_2 \in \mathbf{K}[x,y]$ be coprime and $y$-primitive polynomials such that $0 < \deg_y V_2 \leq \deg_y V_1$.

Using the polynomial division we get a sequence of $y$-primitive polynomials $V_3, \ldots,$ $V_{n+1}$ of decreasing $y$-degrees $0 < \deg_y V_{n+1} < \cdots < \deg_y V_3 < deg_y V_2$ such that

$$\begin{aligned} u_1V_1 &= Q_1V_2 + v_1V_3, \\ u_2V_2 &= Q_2V_3 + v_2V_4, \\ &\vdots \\ u_{n-1}V_{n-1} &= Q_{n-1}V_n + v_{n-1}V_{n+1}, \\ u_nV_n &= Q_nV_{n+1} + v_n, \end{aligned}$$

where $u_1,\ldots,\ u_n,\ v_1,\ldots,\ v_n$ are non-zero polynomials of the ring $\mathbf{K}[x]$. Let be $V_{n+2} = 1$ and write the above equalities in the form

$$(1)_i \qquad u_iV_i = Q_iV_{i+1} + v_iV_{i+2} \quad \text{for } i = 1, \ldots, n.$$

In what follows we call $n$ the number of steps performed by the Euclidean algorithm on input $(V_1,\ V_2)$. We will keep the above notation in all this note.

## 2. Labatie's elimination

Let us define two sequences $d_1, \ldots, d_n$ and $w_1, \ldots, w_n$ of polynomials in $x$ determined by the sequences $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ in a recurrent way. We let $d_1 = \gcd(u_1, v_1)$, $w_1 = \frac{u_1}{d_1}$ and $d_i = \gcd(w_{i-1}u_i, v_i)$, $w_i = \frac{w_{i-1}u_i}{d_i}$ for $i \in \{2, \ldots, n\}$. It is easy to see that $w_i = \frac{u_1\cdots u_i}{d_1\cdots d_i}$ in $\mathbf{K}[x]$ for all $i \in \{1, \ldots, n\}$.

For any $V$, $W \in \mathbf{K}[x,y]$ we let $\{V = 0, W = 0\} = \{\, P \in \mathbf{K}^2 : V(P) = W(P) = 0\,\}$.

**Theorem 2.1** (Labatie 1835)**.** *With notations and assumptions given above we have*

$$\{V_1 = 0, V_2 = 0\} = \bigcup_{i=1}^{n} \left\{ V_{i+1} = 0, \frac{v_i}{d_i} = 0 \right\}.$$

We present the proof of the above theorem in Section 4.

Labatie's theorem shows that the system of equations $V_1(x,y) = 0$, $V_2(x,y) = 0$ is equivalent to the collection of triangular systems

$$V_{i+1}(x,y)=0,\ \frac{v_i}{d_i}(x)=0 \qquad (i=1,\dots,n).$$

Labatie's theorem fell into oblivion for a longtime. At the beginning of the 1990's Lazard in [La] proved that every system of polynomial equations in many unknowns with a finite number of solutions in the algebraic closure of $\mathbf{K}$ is equivalent to the union of triangular systems, which can be obtained from Gröbner bases. Kalkbrener in [Kalk1] and [Kalk2] developed the theory of elimination sequences based on the Euclidean algorithm. His method of computing solutions of systems of polynomials equations turned out to be very efficient if applied to systems of two or three unknowns (see [Kalk2] and the references given therein for the comparison with Gröbner basis methods). Neither Lazard nor Kalkbrener mentioned Labatie's work. Only Glashof in [Glas] recalled Labatie's method after Netto [Ne] and compared it with Kalkbrener's approach to polynomials equations. In what follows we need the notion of multiplicity of a solution of a system of two equations in two unknowns. The definition we are going to present is quite sophisticated. The reader not acquainted with it may assume the five properties of multiplicity given below as axiomatic definition of this notion.

Let $P\in\mathbf{K}^2$. We define the local ring of rational functions regular at $P$ to be

$$\mathbf{K}[x,y]_P=\left\{\frac{R}{S}\ :\ R,S\in\mathbf{K}[x,y], S(P)\neq 0\right\}.$$

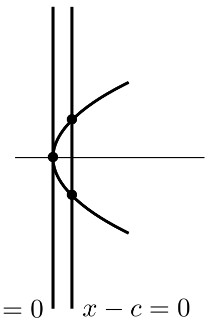
The ring $\mathbf{K}[x,y]_P$ is a unique factorization domain. The units of $\mathbf{K}[x,y]_P$ are rational functions $\frac{R}{S}$ such that $R(P)S(P)\neq 0$.

Let $(V,W)_P$ be the ideal generated by polynomials $V$ and $W$ in $\mathbf{K}[x,y]_P$. Following [Ful], we define the *intersection multiplicity* $i_P(V,W)$ to be the dimension of the $\mathbf{K}$-vector space $\mathbf{K}[x,y]_P/(V,W)_P$. We call also $i_P(V,W)$ the *multiplicity of the solution* $P$ of the system $V=0$, $W=0$.

Let us recall the basic properties of the intersection multiplicity which hold for any field $\mathbf{K}$ (not necessarily algebraically closed):

(1) $i_P(V,W)<+\infty$ if and only if $P\notin\{\gcd(V,W)=0\}$,
(2) $i_P(V,W)>0$ if and only if $P\in\{V=W=0\}$,
(3) $i_P(V,WW')=i_P(V,W)+i_P(V,W')$,
(4) $i_P(V,W)$ depends only on the ideal $(V,W)_P$.
Intuitively: $i_P(V,W)$ does not change when we replace the system $V=0$, $W=0$ by another one equivalent to it near $P$.
Moreover, it is easy to check that
(5) if $P=(a,b)$ is a solution of the triangular system $W(x,y)=0$, $w(x)=0$ then $i_P(W,w)=(\mathrm{ord}_a w)(\mathrm{ord}_b W(a,y))$, where $\mathrm{ord}_c p$ denotes the multiplicity of the root $c$ in the polynomial $p=p(x)\in\mathbf{K}[x]$. By convention $\mathrm{ord}_c p=0$ if $p(c)\neq 0$.

The following example may be helpful to acquire an intuition of intersection multiplicity. Let us consider the parabola $y^2 - x = 0$ over the field of real numbers. Applying Property 5 to the triangular system $y^2 - x = 0$, $x - c = 0$ we check that the axis $x = 0$ intersects the parabola in $(0, 0)$ with multiplicity 2 but the line $x - c = 0$, where $c > 0$ intersects it in two points $(c, \sqrt{c})$ and $(c, -\sqrt{c})$, each with multiplicity 1. If $c \to 0^+$ then the two points coincide.




Note also that the system of equations $y^2 - x = 0$, $x - c = 0$ has for $c \neq 0$ two complex solutions, which are arbitrary close to the origin for small enough complex $c$. This observation leads to the *dynamic definition* of intersection multiplicity for algebraic complex curves (see [Te], Section 6).

The following theorem due to Bonnet [Bo] is an improvement of Labatie's result:

**Theorem 2.2** (Bonnet 1847). *For any $P \in \mathbf{K}^2$ we have*

$$i_P(V_1, V_2) = \sum_{i=1}^{n} i_P\Big(V_{i+1}, \frac{v_i}{d_i}\Big).$$

Bonnet, like Labatie, considered polynomials with complex coefficients and used the definition of the intersection multiplicity in terms of Puiseux series. In Section 5 we present a short proof of Theorem 2.2 based on Labatie's calculations (Section 3) and the properties of the intersection multiplicity listed above.

**Example 2.3.** *Let $V_1 = y^5 - x^3$, $V_2 = y^3 - x^4$. Using the Euclidean algorithm we get $y^5 - x^3 = y^2(y^3 - x^4) + x^3(xy^2 - 1)$, $x(y^3 - x^4) = y(xy^2 - 1) + y - x^5$ and $xy^2 - 1 = (xy + x^6)(y - x^5) + x^{11} - 1$. Hence we have $(u_1, u_2, u_3) = (1, x, 1)$, $(v_1, v_2, v_3) = (x^3, 1, x^{11} - 1)$ and $(d_1, d_2, d_3) = (1, 1, 1)$. By Labatie's theorem, we get*

$$\{y^5 - x^3 = 0, y^3 - x^4 = 0\} =$$
$$\{y^3 - x^4 = 0, x^3 = 0\} \cup \{xy^2 - 1 = 0, 1 = 0\} \cup \{y - x^5 = 0, x^{11} - 1 = 0\}.$$

*Therefore the systems $V_1 = 0$, $V_2 = 0$ has two solutions $(0, 0)$ and $(1, 1)$ in $\mathbf{K}$ and ten solutions in the algebraic closure of $\mathbf{K}$. To compute the multiplicities of the* *solutions we use Bonnet's theorem:*

$i_0(y^5-x^3, y^3-x^4) = i_0(y^3-x^4, x^3)+i_0(xy^2-1, 1)+i_0(y-x^5, x^{11}-1) = 3\cdot 3+0+0 = 9.$

*The remaining multiplicities are equal to one. Thus the system* $V_1 = 0$, $V_2 = 0$ *has* $9 + 11 = 20$ *solutions counted with multiplicities.*

## 3. Auxiliary lemmas

Recall that the polynomials $w_i$ and $\frac{v_i}{d_i}$ are coprime.

**Lemma 3.1.** *There exist two sequences of polymomials* $G_0,\dots, G_n$ *and* $H_0,\dots, H_n$ *in the ring* $\mathbf{K}[x, y]$ *such that*

$$(2)_i \qquad w_{i-1}V_1 = G_{i-1}V_i + G_{i-2}V_{i+1}\frac{v_{i-1}}{d_{i-1}},$$

$$(3)_i \qquad w_{i-1}V_2 = H_{i-1}V_i + H_{i-2}V_{i+1}\frac{v_{i-1}}{d_{i-1}}$$

*for* $i \in \{2,\dots, n + 1\}$.

**Proof.** We proceed by induction on $i$. Let's check the first identity. From the equality $u_1V_1 = Q_1V_2 + v_1V_3$ it follows that $d_1 = \gcd(u_1, v_1)$ divides the product $Q_1V_2$ and consequently the polynomial $Q_1$ since $V_2$ is $y$-primitive. Letting $G_0 = 1$, $G_1 = \frac{Q_1}{d_1}$ we get $w_1V_1 = G_1V_2+G_0V_3\frac{v_1}{d_1}$ that is $(2)_2$. Suppose now that $2 \le i < n+1$ and that for some polynomials $G_{i-1}$ and $G_{i-2}$ the identity $(2)_i$ holds. Multiplying the identity $(2)_i$ by the polynomial $u_i$ we get

$$w_{i-1}u_iV_1 = u_iG_{i-1}V_i + u_iG_{i-2}V_{i+1}\frac{v_{i-1}}{d_{i-1}}.$$

Let us insert to the identity above $u_iV_i = Q_iV_{i+1} + v_iV_{i+2}$. After simple computations we get:

$$w_{i-1}u_iV_1 = \Big(G_{i-1}Q_i + u_iG_{i-2}\frac{v_{i-1}}{d_{i-1}}\Big)V_{i+1} + G_{i-1}v_iV_{i+2}.$$

Since $d_i = \gcd(w_{i-1}u_i, v_i)$ and the polynomial $V_{i+1}$ is $y$-primitive we get that $G_i := \frac{G_{i-1}Q_i}{d_i} + G_{i-2}\frac{u_iv_{i-1}}{d_id_{i-1}}$ is a polynomial and we have

$$w_iV_1 = G_iV_{i+1} + G_{i-1}V_{i+2}\frac{v_i}{d_i},$$

which is the identity $(2)_{i+1}$. This proves the first part of the lemma.

To prove the identity $(3)_i$ note that

$$w_1V_2 = H_1V_2 + H_0V_3\frac{v_1}{d_1}$$

if we let $H_0 = 0$ and $H_1 = \frac{u_1}{d_1}$. This proves $(3)_2$. To check $(3)_i$ we proceed analogously to the proof of $(2)_i$ : it suffices to replace $G_i$ by $H_i$. ■

**Remark 3.2.** *The polynomials $G_i$ are defined by $G_0 = 1$, $G_1 = \frac{Q_1}{d_1}$, $G_i = \frac{G_{i-1}Q_i}{d_i} + \frac{G_{i-2}u_i v_{i-1}}{d_{i-1}d_i}$ and the polynomials $H_i$ by $H_0 = 0$, $H_1 = \frac{u_1}{d_1}$ and $H_i = \frac{H_{i-1}Q_i}{d_i} + \frac{H_{i-2}u_i v_{i-1}}{d_{i-1}d_i}$.*

**Lemma 3.3.** *With the notations of Lemma 3.1 we have the identities*

$$(4)_i \qquad (-1)^i \frac{v_1 \cdots v_{i-1}}{d_1 \cdots d_{i-1}} V_{i+1} = H_{i-1}V_1 - G_{i-1}V_2 \qquad \textit{for } i \in \{2, \ldots, n+1\}.$$

**Proof.** Let $D_i = G_i H_{i-1} - G_{i-1}H_i$ for $i \in \{2, \ldots, n\}$. Consider the system of equations $(2)_i$, $(3)_i$ as a linear system with unknowns $V_i$, $V_{i+1}\frac{v_{i-1}}{d_{i-1}}$ with determinant equals $D_{i-1}$. Using Cramer's rule we get

$$\begin{aligned} D_{i-1}V_i &= w_{i-1}\left(H_{i-2}V_1 - G_{i-2}V_2\right), \\ D_{i-1}V_{i+1}\frac{v_{i-1}}{d_{i-1}} &= -w_{i-1}(H_{i-1}V_1 - G_{i-1}V_2). \end{aligned}$$

Replacing in the first equality $i$ by $i+1$ we obtain

$$(1) \qquad D_i V_{i+1} = w_i(H_{i-1}V_1 - G_{i-1}V_2).$$

Multiplying the second equality by $\frac{u_i}{d_i}$ we get

$$(2) \qquad D_{i-1}V_{i+1}\frac{v_{i-1}}{d_{i-1}}\frac{u_i}{d_i} = -w_i(H_{i-1}V_1 - G_{i-1}V_2).$$

Comparing the left sides of (1) and (2) and cancelling $V_{i+1}$ we have $D_i = -\frac{v_{i-1}u_i}{d_{i-1}d_i}D_{i-1}$. Moreover $D_1 = G_1H_0 - G_0H_1 = -\frac{u_1}{d_1}$ and by induction we have

$$D_i = (-1)^i w_i \frac{v_1 \cdots v_{i-1}}{d_1 \cdots d_{i-1}}$$

which inserted into formula (1) gives the identity $(4)_i$. ∎

## 4. Proof of Labatie's theorem

We can now give the proof of Theorem 2.1: fix a point $P \in \mathbf{K}^2$. If $V_i(P) = \frac{v_{i-1}}{d_{i-1}}(P) = 0$ for a value $i \in \{2, \ldots, n+1\}$ then from Lemma 3.1 it follows that $V_1(P) = V_2(P) = 0$ given that $w_{i-1}(P) \neq 0$ since $w_{i-1}$, $\frac{v_{i-1}}{d_{i-1}}$ are coprime.

Suppose now that $V_1(P) = V_2(P) = 0$. From the identity $(4)_{n+1}$ of Lemma 3.3 we get $\frac{v_1 \cdots v_n}{d_1 \cdots d_n}(P) = 0$. Therefore at least one of polynomials $\frac{v_1}{d_1}, \ldots, \frac{v_n}{d_n}$ vanishes at $P$. If $\frac{v_1}{d_1}(P) = 0$ then $P \in \{V_2 = \frac{v_1}{d_1} = 0\}$.

If the smallest index $i$ for which $\frac{v_i}{d_i}(P) = 0$ is strictly greater than 1 then we get, by the identity $(4)_i$ , that $V_{i+1}(P) = 0$ because $\frac{v_1 \cdots v_{i-1}}{d_1 \cdots d_{i-1}}(P) \neq 0$ by the definition of $i$. This proves the theorem.

## 5. Proof of Bonnet's theorem

Fix a point $P \in \mathbf{K}^2$. If $\frac{v_1 \cdots v_n}{d_1 \cdots d_n}(P) \neq 0$ then by $(4)_{n+1}$ we get

$$1 \in (V_1, V_2)_P \tag{3}$$

which implies $i_P(V_1, V_2) = 0$.

On the other hand we have $i_P\left(V_{i+1}, \frac{v_i}{d_i}\right) = 0$ since $\frac{v_i}{d_i}(P) \neq 0$ for $i \in \{1, \dots, n\}$ and the theorem holds in the case under consideration.

Suppose now that $\frac{v_1 \cdots v_n}{d_1 \cdots d_n}(P) = 0$ and let $i_0$ be the smallest index $i \in \{1, \dots, n\}$ such that $\frac{v_{i_0}}{d_{i_0}}(P) = 0$. Therefore we have $w_{i_0}(P) \neq 0$ since $\frac{v_{i_0}}{d_{i_0}}$ and $w_{i_0}$ are coprime. Let us check that

$$(V_1, V_2)_P = \left(V_{i_0+1}, V_{i_0+2}\frac{v_{i_0}}{d_{i_0}}\right)_P. \tag{4}$$

From $(2)_{i_0+1}$ and $(3)_{i_0+1}$ we get

$$V_1, V_2 \in \left(V_{i_0+1}, V_{i_0+2}\frac{v_{i_0}}{d_{i_0}}\right)_P. \tag{5}$$

On the other hand, from $(4)_{i_0}$ (if $i_0 > 1$, the case $i_0 = 1$ being obvious), we obtain

$$V_{i_0+1} \in (V_1, V_2)_P \tag{6}$$

and from $(4)_{i_0+1}$, we have

$$\frac{v_{i_0}}{d_{i_0}} V_{i_0+2} \in (V_1, V_2)_P. \tag{7}$$

Combining (5), (6) and (7) we get (4). Equality (4) and the additive property of intersection multiplicity give

$$i_P(V_1, V_2) = i_P\left(V_{i_0+1}, \frac{v_{i_0}}{d_{i_0}}\right) + i_P(V_{i_0+1}, V_{i_0+2}). \tag{8}$$

If $i_0 = n$ then (8) reduces to

$$i_P(V_1, V_2) = i_P\left(V_{n+1}, \frac{v_n}{d_n}\right) \tag{9}$$

since $V_{n+2} = 1$.

To prove Theorem 2.2 we shall proceed by induction on the number $n$ of steps performed by the Euclidean algorithm. For $n = 1$ the theorem follows from (9) since $n = 1$ implies $i_0 = 1$. Let $n > 1$ and suppose that the theorem holds for all pairs of polynomials for which the number of steps performed by the Euclidean algorithm is strictly less than $n$.

We assume that $i_0 < n$ since for $i_0 = n$ the theorem is true by (9).

Let us put $\overline{V}_j = V_{i_0+j}$, where $j \in \{1, 2, \ldots, n - i_0 + 2\}$. The number of steps performed by the Euclidean algorithm on input $(\overline{V}_1, \overline{V}_2)$ is equal to $\overline{n} = n - i_0 < n$. We have $\overline{u}_j = u_{i_0+j}$ and $\overline{v}_j = v_{i_0+j}$ for $j \in \{1, \ldots, \overline{n}\}$. To relate $\overline{d}_j$ and $d_{i_0+j}$ we introduce some notation. We will write $u \sim \tilde{u}$ for polynomials $u, \tilde{u}$ associated in the local ring $\mathbf{K}[x, y]_P$. If $u, \tilde{u} \in \mathbf{K}[x]$ then $u \sim \tilde{u}$ if and only if there exist polynomials $r, s \in \mathbf{K}[x]$ such that $su = r\tilde{u}$ and $r(P)s(P) \neq 0$. Note that $\gcd(u, v) \sim \gcd(\tilde{u}, v)$ if $u \sim \tilde{u}$. We claim that

$$\overline{d}_j \sim d_{i_0+j}, \ \ \overline{w}_j \sim w_{i_0+j} \ \text{ for } \ j \in \{1, \ldots, \overline{n}\}. \tag{10}$$

Let us check (10) by induction on $j$.

If $j = 1$ then $\overline{d}_1 = \gcd(\overline{u}_1, \overline{v}_1) = \gcd(u_{i_0+1}, v_{i_0+1}) \sim \gcd(w_{i_0} u_{i_0+1}, v_{i_0+1}) = d_{i_0+1}$ since $w_{i_0} \sim 1$. Hence we get $\overline{w}_1 = \frac{\overline{u}_1}{\overline{d}_1} = \frac{u_{i_0+1}}{\overline{d}_1} \sim \frac{w_{i_0} u_{i_0+1}}{d_{i_0+1}}$, which proves (10) for $j = 1$.

Suppose that (10) holds for a $j < \overline{n}$. Then we get

$$\overline{d}_{j+1} = \gcd(\overline{w}_j \overline{u}_{j+1}, \overline{v}_{j+1}) \sim \gcd(w_{i_0+j} u_{i_0+j+1}, v_{i_0+j+1}) = d_{i_0+j+1}$$

since $\overline{w}_j \sim w_{i_0+j}$ by the induction assumption, and

$$\overline{w}_{j+1} = \frac{\overline{w}_j \overline{u}_{j+1}}{\overline{d}_{j+1}} \sim \frac{w_{i_0+j} u_{i_0+j+1}}{d_{i_0+j+1}} = w_{i_0+j+1}.$$

This finishes the proof of (10).

Now we can pass to the proof of the theorem. By the inductive assumption applied to the pair $\overline{V}_1, \overline{V}_2$ we get

$$\begin{aligned} i_P(V_{i_0+1}, V_{i_0+2}) &= i_P(\overline{V}_1, \overline{V}_2) = \sum_{j=1}^{\overline{n}} i_P\left(\overline{V}_{j+1}, \frac{\overline{v}_j}{\overline{d}_j}\right) \\ &= \sum_{j=1}^{\overline{n}} i_P\left(V_{i_0+j+1}, \frac{v_{i_0+j}}{d_{i_0+j}}\right) = \sum_{i=i_0+1}^{n} i_P\left(V_{i+1}, \frac{v_i}{d_i}\right) \end{aligned}$$

since $\overline{d}_j \sim d_{i_0+j}$ by (10) which together with (8) proves the inductive step and so the theorem.

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