SINGULAR POINTS
OF COMPLEX ALGEBRAIC CURVES

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December 2013

These notes are based on my introductory lectures on plane curve singularities given at Institute of Mathematics of Maria Curie-Skłodowska University in autumn 2013.

Contents

1 Plane affine curves and polynomial automorphisms of $\mathbb{C}^2$

1.1 Degree and asymptotic directions

Let $F = F(X,Y) = \sum c_{\alpha\beta} X^\alpha Y^\beta$ be a polynomial with complex coefficients $c_{\alpha\beta}$. We put

$$\operatorname{supp} F = \{(\alpha,\beta) \in \mathbb{N}^2 : c_{\alpha\beta} \neq 0\},$$

$$\deg F = \sup\{\alpha+\beta : (\alpha,\beta) \in \operatorname{supp} F\},$$

$$F^+ = \sum_{\alpha+\beta=\deg F} X^\alpha Y^\beta.$$

By conventions: $\deg 0 = -\infty$, $0^+ = 0$. We have $\deg FG = \deg F + \deg G$, $(FG)^+ = F^+G^+$.

The ring $\mathbb{C}[X,Y]$ of polynomials in two variables is an UFD (unique factorization domain). In particular F is an irreducible polynomial if and only if F is a prime element of $\mathbb{C}[X,Y]$. Using the algorithm of the greatest common divisor (see, for example M. Bôcher, Introduction to Higher Algebra, Chapter XVI) we check

Lemma 1.1 *If $F = F(X,Y)$ and $G = G(X,Y)$ have no common factor of positive degree in Y then $a(X) = A(X,Y)F(X,Y) + B(X,Y)G(X,Y)$ in $\mathbb{C}[X,Y]$ with $a(X) \neq 0$.*

In what follows we identify the polynomial $F = F(X,Y)$ and the polynomial function $\mathbb{C}^2 \ni (x,y) \mapsto F(x,y) \in \mathbb{C}$. We put

$$V(F) = \{(x,y) \in \mathbb{C} : F(x,y) = 0\} .$$

Lemma 1.2

(a) *If $F \neq$ const then $V(F)$ and $\mathbb{C}^2 \setminus V(F)$ are infinite.*

(b) *If $F, G \in \mathbb{C}[X,Y]$ are relatively prime then the set $V(F) \cap V(G)$ is finite.*

(c) *If F has no multiple factors and $V(F) \subset V(G)$ then F divides G.*

We leave to the reader the simple proof of Lemma 1.2 (to check (b) use Lemma 1.1).

A subset $V \subset \mathbb{C}^2$ is an affine plane algebraic curve (an affine curve in short) if there is a polynomial $F \neq$ const such that $V = V(F)$. If F has no multiple factors then we say that F is a minimal polynomial of the affine curve $V = V(F)$. Any two minimal polynomials of the curve V differ only by a constant factor. An affine curve is irreducible if its minimal polynomial is irreducible. Note that $V(FG) = V(F) \cup V(G)$. We check easily

Lemma 1.3

(a) *V is irreducible if and only if there are no two affine curves V' and V'' such that $V = V' \cup V''$ and $V', V'' \neq V$.*

(b) *Let $V \subset \mathbb{C}^2$ be an affine curve. Then V has a decomposition $V = V_1 \cup \ldots \cup V_s$, where each V_i is an irreducible curve and $V_i \not\subset V_j$ for $i \neq j$. This decomposition is unique up to the order in which $V_1, \ldots, V_s$ are written.*

To define the notion of projective line we define an equivalence relation on the set $\mathbb{C}^2 \setminus \{(0,0)\}$ by setting

$$(x,y) \sim (x',y')$$

if there is a nonzero complex number λ such that $x' = \lambda x$ and $y' = \lambda y$. Then we define

$$\mathbb{P}^1(\mathbb{C}) = \mathbb{C}^2 \setminus \{(0,0)\} \Big/ \sim .$$

Let V be an algebraic curve with minimal polynomial $F = F(X,Y)$. Then we put

$\deg V = \deg F$,

$V_\infty = \{(k:l) \in \mathbb{P}^1(\mathbb{C}) : F^+(k,l) = 0\}$ (the set of asymptotic directions of V).

Theorem 1.4 (Geometric characterization of degree)
Let V be an affine curve of degree $n = \deg V$ and let L be a line such that $L \not\subset V$. Then

(a) *$\#(V \cap L) \leq n$,*

(b) *if $V_\infty \cap L_\infty \neq \varnothing$ then $\#(V \cap L) < n$,*

(c) *if $V_\infty \cap L_\infty = \varnothing$ then the set of lines L' parallel to L such that $\#(V \cap L') < n$ is finite.*

Proof of Theorem 1.4
Let F be the minimal polynomial of V. Then $\deg F = n$ and the set V_∞ has the equation $F^+ = 0$. Let $X = a + kT$, $Y = b + lT$ be a parametrization of the line L. Then $L_\infty = \{(k : l)\}$. Let

$$\Phi(T) = F(a + kT, b + lT) \in \mathbb{C}[T] \ .$$

It is easy to check that

(1) $\Phi(T) = F^+(k, l)T^n + \ldots + F(a, b)$,

(2) $\#(V \cap L) = \#\Phi^{-1}(0)$.

One has $\Phi(T) \neq 0$ because we have assumed $L \not\subset V$. From (1) and (2) we get immediately properties (a) and (b) of Theorem 1.4. To check (c) we need

Lemma 1.5 *If $F \in \mathbb{C}[X, Y]$ has no multiple factors and $F^+(k, l) \neq 0$ then the polynomials F and $DF := k(\partial F/\partial X) + l(\partial F/\partial Y)$ are relatively prime.*

Proof of Lemma 1.5
The condition $F^+(k, l) \neq 0$ implies $\deg DF = \deg F - 1$. Let $F = F_1 \ldots F_r$ with irreducible factors F_i. Then $DF = (DF_1)F_2 \ldots F_r + \ldots + F_1 \ldots F_{r-1}DF_r$. Suppose that F and DF have a common factor, we may assume that it is the polynomial F_1. If F_1 divides DF then F_1 divides $(DF_1)F_2 \ldots F_r$ and consequently DF_1 since F_1 and $F_2 \ldots F_r$ are coprime. Clearly F_1^+ divides F^+ and $F_1^+(k, l) \neq 0$. By the remark made at the beginning of the proof $\deg(DF_1) = \deg F_1 - 1$. A contradiction since from the fact that F_1 divides DF_1 it follows that $\deg F_1 \leq \deg(DF_1)$ ∎

Now, we may continue the proof of Theorem 1.4. We have to check property (c) of Theorem 1.4. Let us consider the set $W := V(F) \cap V(k(\partial F/\partial X) + l(\partial F/\partial Y))$. It is finite by Lemmas 1.5 and 1.2 (b). Let $N = \#W$ and let $L_1, \ldots, L_N$ be the lines passing through the points of W and parallel the line L. Let L' be a line parallel to L and different from $L_1, \ldots, L_N$. We claim that $\#(V \cap L') = n$. Let $X = a' + kT$, $Y = b' + lT$ be a parametrization of L' and consider the polynomial

$$F(a' + kT, b' + lT) = F^+(k, l)T^n + \ldots + F(a', b') \ .$$

It is a polynomial of degree n. All roots of this polynomial are simple (i.e. of multiplicity 1). Indeed, if the polynomial $F(a' + kT, b' + lT)$ had a root $t_0 \in \mathbb{C}$ of multiplicity > 1 then the point $(x_0, y_0) = (a' + kt_0, b' + lt_0)$ would lie in the set W, which contradicts the assumption $W \cap L' = \varnothing$. Thus we get $\#(V \cap L') = n$ because different roots of $F(a' + kT, b' + lT)$ correspond to different points of $V \cap L'$ ∎

1.2 Polynomial automorphisms of $\mathbb{C}^2$

A polynomial map $F : \mathbb{C}^2 \to \mathbb{C}^2$ is called polynomial automorphism if $F^{-1} : \mathbb{C}^2 \to \mathbb{C}^2$ exists and is also a polynomial map. We denote by $\mathrm{Aut}(\mathbb{C}^2)$ the group of polynomial automorphisms of $\mathbb{C}^2$. Examples

1) $F(X, Y) = (aX + bY + c, a_1X + b_1Y + c_1)$, $ab_1 - a_1b \neq 0$

2) $F(X, Y) = (aX, bY + P(X))$, $ab \neq 0$, $P(X)$ is a polynomial of degree > 1, or $F(X, Y) = (aX + Q(Y), bY)$, where $Q(Y)$ is a polynomial of degree > 1.

Theorem 1.6 (Jung-Van der Kulk)
Any polynomial automorphism is a composition of automorphisms of type 1) and 2).

Proof. See A. van den Essen, Polynomial Automorphisms and the Jacobian Conjecture. Progress in Mathematics, vol. 190. Birkhäuser 2000.

For any polynomial map $F = (F_1, F_2)$ we set $\operatorname{jac} F = (\partial F_1/\partial X)(\partial F_2/\partial Y) - (\partial F_1/\partial Y)(\partial F_2/\partial X)$. Let $F : \mathbb{C}^2 \to \mathbb{C}^2$ be a polynomial automorphism. Then $F^{-1} \circ F =$ identity and

$$((\operatorname{jac} F)^{-1} \circ F)(\operatorname{jac} F) = 1 \text{ in } \mathbb{C}[X, Y] \ .$$

Thus $\operatorname{jac} F = \text{const} \neq 0$.

Jacobian Conjecture (Keller 1939)
Let $F : \mathbb{C}^2 \to \mathbb{C}^2$ *be a polynomial map such that* $\operatorname{jac} F = \text{const} \neq 0$. *Then* F *is a polynomial automorphism.*

This conjecture is still open, see S. Smale, Mathematical Problems for the Next Century. Mathematical Intelligencer, vol. 20 (Springer 1998), pp. 7–15.

Polynomial curves and automorphisms

An irreducible curve $V \subset \mathbb{C}^2$ is a polynomial curve if there exists a pair of polynomials $P(T), Q(T) \in \mathbb{C}[T]$, in one variable T, such that

(i) $(x, y) \in V$ if and only if there exists $t \in \mathbb{C}$ such that $x = P(t)$, $y = Q(t)$.

(ii) If $t_1 \neq t_2$ then $(P(t_1), Q(t_1)) \neq (P(t_2), Q(t_2))$.

Theorem 1.7 *Let* V *be a polynomial curve with parametrization* $P(T) = aT^n + \ldots$, $Q(T) = bT^n + \ldots$, *where* $a \neq 0$ *or* $b \neq 0$ *then* $\deg V = n$ *and* $V_\infty = \{(a : b)\}$.

Proof. We may assume that $\deg P < \deg Q$ (the case $\deg P = \deg Q$ we treat analogously). Then $a = 0$ and $b \neq 0$, so $(a : b) = (0 : 1)$. To calculate $\deg V$ and V_∞ let us consider a line L such that $L \not\subset V$. If $L = V(pX + qY + r)$, where $p, q, r \in \mathbb{C}$, $p \neq 0$ or $q \neq 0$ then $\#(V \cap L) = \#\{t \in \mathbb{C} : p\,P(t) + q\,Q(t) + r = 0\}$. Since $L \not\subset V$ the polynomial $p\,P(t) + q\,Q(t) + r$ is nonzero of degree $\leq \deg Q = n$, the equality holding if and only if $q \neq 0$. We get then

(3) $\#(V \cap L) \leq n$,

(4) if $L_\infty = \{(0 : 1)\}$ then $\#(V \cap L) < n$.

To finish the proof it suffices to check

(5) if $L_\infty \neq \{(0 : 1)\}$ then there exists a line L' parallel to L such that $\#(V \cap L') = n$.

To check (5) observe that $L_\infty \neq \{(0 : 1)\}$ if and only if $q \neq 0$. Suppose $q \neq 0$ and let $t_1, \ldots, t_s$ $(s \leq n)$ be all roots of the polynomial $p(dP/dT) + q(dQ/dT)$. Let $r' \in \mathbb{C}$ be such that $p\,P(t_j) + q\,Q(t_j) \neq r'$ for $j = 1, \ldots, s$. Let $L' = V(pX + qY = r')$. Then $\#(V \cap L') = n$ since the polynomial $p\,P(T) + q\,Q(T) + r'$ is of degre n and has no multiple roots. Now Theorem 1.7 follows from Theorem 1.4 ∎

Let us present some applications to polynomial automorphisms. The following lemma is an immediate consequence of definitions.

Lemma 1.8 *Let $F = (F_1, F_2) : \mathbb{C}^2 \to \mathbb{C}^2$ be a polynomial automorphism and let $F^{-1} = (G_1, G_2)$. Then $V(F_1)$ and $V(F_2)$ are polynomial curves with parametrizations $(G_1(0, Y_2), G_2(0, Y_2))$ and $(G_1(Y_1, 0), G_2(Y_1, 0))$ respectively.*

For any polynomial automorphism $F = (F_1, F_2)$ we set $\deg F = \max(\deg F_1, \deg F_2)$.

Theorem 1.9 *For any $F \in \mathrm{Aut}(\mathbb{C}^2)$: $\deg F^{-1} = \deg F$.*

Proof. We have $\deg F = max(\deg F_1, \deg F_2) = \max(\deg V(F_1), \deg V(F_2))$. By Theorem 1.7 we get $\deg V(F_1) = \max(\deg G_1(0, Y_2), \deg G_2(0, Y_2)) \leq \max(\deg G_1, \deg G_2) = \deg F^{-1}$ and $\deg V(F_2) = \max(\deg G_1(Y_1, 0), \deg G_2(Y_1, 0)) \leq \max(\deg G_1, \deg G_2) = \deg F^{-1}$. Therefore $\deg F \leq \deg F^{-1}$. Applying the inequality $\deg F \leq \deg F^{-1}$ to the automorphism F^{-1} we get $\deg F^{-1} \leq \deg(F^{-1})^{-1} = \deg F$. Consequently $\deg F = \deg F^{-1}$ ■

Theorem 1.10 (Jelonek). *Let $F \neq \mathrm{id}$ be a polynomial automorphism. If an affine curve $V \subset \mathbb{C}^2$ lies in the set $\mathrm{Fix}\, F = \{(x, y) \in \mathbb{C}^2 : F(x, y) = (x, y)\}$ then $\#V_\infty = 1$.*

Proof. We may assume that $\deg V > 1$. Then $\deg F_1 > 1$ and $\deg F_2 > 1$. We have $V \subset \{(x, y) \in \mathbb{C}^2 : F_1(x, y) - x = 0\} = V(F_1 - X)$. The curve $V(F_1 - X)$ has one asymptotic direction since $\deg F_1 > 1$ and $V(F_1)$ being polynomial curve has one asymptotic direction by Theorem 1.7. Thus $\#V_\infty = 1$ ■

Theorem 1.11 (Jelonek). *Let $W \subset \mathbb{C}^2$ be an affine curve with at least two asymptotic directions. Suppose that $F, \tilde{F} \in \mathrm{Aut}(\mathbb{C}^2)$ and $F|W = \tilde{F}|W$. Then $F = \tilde{F}$.*

Proof. The condition $F|W = \tilde{F}|W$ implies that $W \subset \mathrm{Fix}(\tilde{F}^{-1} \circ F)$. Since $\#W \geq 2$ we get $\tilde{F}^{-1} \circ F = \mathrm{id}$ by Theorem 1.10 and $F = \tilde{F}$ ■

The reader will find more result of this type in Z. Jelonek, Sets determining polynomial automorphisms of $\mathbb{C}^2$. Bull. Polish Acad. Sci. Math. 37, 1989.

2 Singular points and tangents

Let $F = F(X, Y) \in \mathbb{C}[X, Y]$ be a nonconstant polynomial. For any $p = (a, b) \in \mathbb{C}^2$ we write

$$F = \sum c^{(p)}_{\alpha,\beta} (X - a)^\alpha (Y - b)^\beta .$$

We define

$$\mathrm{ord}_p F = \inf\{\alpha + \beta : c^{(p)}_{\alpha,\beta} \neq 0\} \text{ (the order of } F \text{ at } p)$$

and

$$\mathrm{in}_p F = \sum_{\alpha+\beta=\mathrm{ord}_p F} c^{(p)}_{\alpha,\beta} (X - a)^\alpha (Y - b)^\beta \text{ (the initial form of } F \text{ at } p) .$$

We say that F is homogeneous in $X - a$ and $Y - b$ if $F = \mathrm{in}_p F$ for $p = (a, b)$.

Let $V \subset \mathbb{C}^2$ be an affine curve with minimal polynomial $F = F(X, Y)$. We put $\mathrm{ord}_p V = \mathrm{ord}_p F$ (the multiplicity of V at p). The lines passing through p given by the equation $\mathrm{in}_p F = 0$ are called tangents to V at p. Thus the curve V has at most $\mathrm{ord}_p V$ tangents at p. We put $C_p(V) = V(\mathrm{in}_p F)$ (the tangent cone to V at p).

Theorem 2.1 (Geometrical characterization of multiplicity)
Let V be an affine curve of degree $n > 0$ and let $m = \mathrm{ord}_p V$ be the multiplicity of V at a given point $p \in V$. Suppose that all components of V passing through p are of degree > 1. Let L be a line passing through p. Then we have

(a) $\#((V \setminus \{p\}) \cap L) \leq n - m$,

(b) $\#((V \setminus \{p\}) \cap L) < n - m$ *if* $L \in C_p(V)$,

(c) *the set of lines* L *passing through* p *such that* $\#((V \setminus \{p\}) \cap L) < n - m$ *is finite.*

Proof
Let $F = F(X, Y)$ be the minimal polynomial of V. For any line $L : X = a + kT$, $Y = b + lT$ passing through $p = (a, b)$ we consider $\Phi_L(T) = F(a + kT, b + lT)$. It is easy to see that $\Phi_L(T) = F^{+}(k, l)T^n + \ldots + (\mathrm{in}_p F)(k, l)T^m$. Observe that

- $\mathrm{ord}_0 \Phi_L \geq m$ with equality if and only if $L \notin C_p(V)$,
- $\#((V \setminus \{p\}) \leq$ the number of nonzero roots of $\Phi_L(T) = 0 \ \leq n - m$, the equality is strict if $\mathrm{ord}_0 \Phi > m$.

This proves properties (a) and (b). To prove (c) we need the following lemma.

Lemma 2.2 *Suppose that* $F = F(X, Y)$ *is of degree* $n > 1$ *and has no multiple factors. Let* $p = (a, b) \in V(F)$ *and suppose that there is no line* L *such that* $p \in L \subset V(F)$. *Then* F *and* $D_p F := (X - a)\frac{\partial F}{\partial X} + (Y - b)\frac{\partial F}{\partial Y}$ *are relative prime.*

We leave the proof of Lemma 2.2 to the reader (cf. the proof of Lemma 1.5).

We are now able to prove property (c). Note that by Lemma 2.2 the set $V(F) \cap V(D_p F)$ is finite. Let us consider the lines passing though p and satisfying the conditions:

- $V_\infty \cap L_\infty = \varnothing$,
- $L \notin C_p(V)$,
- $L \cap V(F) \cap V(D_p F) = \{p\}$.

Obviously all but finite number of lines passing through p satisfy the above conditions. From the first two conditions we infer that the polynomial $\Phi_L(T)$ is of degree n and has $n - m$ nonzero roots counted with multiplicities. We have to check that the third condition implies that nonzero roots of $\Phi_L(T)$ are simple. To this purpose suppose that $\Phi_L(t_0) = 0$ and $(d\Phi_L/dT)(t_0) = 0$ for a $t_0 \in \mathbb{C}$. Let $x_0 = a + kt_0$, $y_0 = b + lt_0$. Then, we have $F(x_0, y_0) = 0$, $k\frac{\partial F}{\partial X}(x_0, y_0) + l\frac{\partial F}{\partial Y}(x_0, y_0) = 0$ and consequently $(x_0, y_0) \in L \cap V(F) \cap V(D_p F) = \{(a, b)\}$ which implies $t_0 = 0$. This proves (c) ■

As an example of the above characterization of multiplicity we will deduce the following

Theorem 2.3 *Let* V *be a polynomial curve with parametrization* $(P(T), Q(T))$ *and* $p = (P(t_0), Q(t_0)) \in V$. *Then* $\mathrm{ord}_P V = \min(\mathrm{ord}_{t_0} P, \mathrm{ord}_{t_0} Q)$.

Proof. We may assume that $t_0 = 0$, $p = (P(0), Q(0)) = (0, 0)$ and $\deg V > 1$. By Theorem 1.7 we have that $n := \deg V = \max(\deg P, \deg Q)$. Let $m = \min(\mathrm{ord}_0 P, \mathrm{ord}_0 Q)$. To check that $\mathrm{ord}_0 V = m$ it sufficies to prove that for any line L passing through 0 we have $\#((V \setminus \{0\}) \cap L) \leq n - m$ with equality for all but finite number of lines passing through 0.

Let L be a line with equation $lX - kY = 0$. The points of the set $(V \setminus \{0\}) \cap L$ correspond to the nonzero roots of the equation $l\,P(T) - k\,Q(T) = 0$. Therefore $\#((V \setminus \{0\}) \cap L) \leq \deg(l\,P(T) - k\,Q(T)) - \mathrm{ord}_0(l\,P(T) - k\,Q(T)) \leq \max(\deg P, \deg Q) - \min(\deg P, \deg Q) = \deg V - m$. To finish the proof we have to show that for all but finite number of lines $L = V(lX - kY)$ the equation $l\,P(T) - k\,Q(T) = 0$ has exactly $n - m$ nonzero roots.

The reader will check

Lemma 2.4 *Suppose that the polynomials $P(T), Q(T) \in \mathbb{C}[T]$ are linearly independent. Then $P(T)Q'(T) - P'(T)Q(T) \neq 0$ in $\mathbb{C}[T]$.*

Since $\deg V > 1$ the polynomials $P(T)$ and $Q(T)$ are linearly independent and the set $\{t \in \mathbb{C} : P(t)Q'(t) - P'(t)Q(t) = 0\}$ is finite. Now, it suffices to observe that for all but a finite number of lines $L = V(lX - kY)$ the following conditions hold

- $\deg(l\,P(T) - k\,Q(T)) = n$,
- $\mathrm{ord}_0(l\,P(T) - k\,Q(T)) = m$.
- Let $t_1, \ldots, t_s$ be nonzero roots of the equation $P(T)Q'(T) - P'(T)Q(T) = 0$. Then L does not pass through the points $(P(t_1), Q(t_1))$, …, $(P(t_s), Q(t_s))$.

Let V be an affine curve with minimal polynomial $F = F(X, Y)$. A point $p = (a, b) \in V$ is a singular point of V if

$$\frac{\partial F}{\partial X}(a, b) = \frac{\partial F}{\partial Y}(a, b) = 0\ .$$

Obviously p is singular $\Leftrightarrow$ $\mathrm{ord}_p V > 1$.

Proposition 2.5 *The set of singular points* $\mathrm{Sing}\, V$ *of the curve* V *is finite.*

Proof. Clearly $\mathrm{Sing}\, V \subset V(F) \cap V(k\frac{\partial F}{\partial X} + l\frac{\partial F}{\partial Y})$. Use Lemma 1.5 ■

Proposition 2.6 *If a polynomial curve V has a regular parametrization $(P(T), Q(T))$ i.e. $(P'(t), Q'(t)) \neq (0, 0)$ for $t \in \mathbb{C}$ then $\mathrm{Sing}\, V = \varnothing$.*

Proof. Use Theorem 2.3 ■

To end with let us quote

Theorem 2.7 (Lin-Zaidenberg)
A polynomial curve has at most one singular point.

Proof. See V. Lin and M. Zaindenberg, An irreducible simply connected algebraic curve in $\mathbb{C}^2$ is equivalent to a quasihomogeneous curve. Dokl. Akad. Nauk SSSR 271(1983), 1048-1052.

3 Formal power series

A formal power series in two variables X, Y is an expression of the form $\sum c_{\alpha\beta} X^\alpha Y^\beta$ where $c_{\alpha\beta} \in \mathbb{C}$ for $\alpha, \beta = 0, 1, \ldots$. The set of all power series in X, Y is denoted by $\mathbb{C}[[X, Y]]$. We define

$$\begin{array}{ll} \text{addition,} & \sum a_{\alpha\beta} X^\alpha Y^\beta + \sum b_{\alpha\beta} X^\alpha Y^\beta = \sum (a_{\alpha\beta} + b_{\alpha\beta}) X^\alpha Y^\beta, \\ \text{scalar mult.,} & a \sum a_{\alpha\beta} X^\alpha Y^\beta = \sum (a\, a_{\alpha\beta}) X^\alpha Y^\beta, \\ \text{multiplication,} & \left(\sum a_{\alpha\beta} X^\alpha Y^\beta\right)\left(\sum b_{\alpha\beta} X^\alpha Y^\beta\right) = \sum_{\alpha,\beta} \Big(\sum_{\substack{\alpha_1+\alpha_2=\alpha \\ \beta_1+\beta_2=\beta}} a_{\alpha_1\beta_1} b_{\alpha_2\beta_2} \Big) X^\alpha Y^\beta. \end{array}$$

With these operators $\mathbb{C}[[X, Y]]$ becomes a $\mathbb{C}-$algebra that includes the polynomial algebra $\mathbb{C}[X, Y]$. Let $F(X, Y) = \sum c_{\alpha\beta} X^\alpha Y^\beta \in \mathbb{C}[[X, Y]]$. We put

$F(0, 0) = c_{00}$,

$$\operatorname{ord} F = \inf\{\alpha+\beta : c_{\alpha\beta} \neq 0\},$$

$$\operatorname{in} F = \sum_{\alpha+\beta=\operatorname{ord} F} c_{\alpha\beta} X^\alpha Y^\beta .$$

By conventions: $\operatorname{ord} 0 = +\infty$, $\operatorname{in} 0 = 0$. We have $\operatorname{ord}(F+G) \geq \inf\{\operatorname{ord} F, \operatorname{ord} G\}$ with equality if $\operatorname{ord} F \neq \operatorname{ord} G$ and $\operatorname{ord} FG = \operatorname{ord} F + \operatorname{ord} G$. The ring $\mathbb{C}[[X,Y]]$ has no zero divisors. A power series $U = U(X,Y)$ is a unit in $\mathbb{C}[[X,Y]]$ (i.e. $UV = 1$ for a power series $V = V(X,Y)$) if and only if $U(0,0) \neq 0$. We write $F \sim G$ if there is a unit U such that $G = UF$. If $G = G(X_1,Y_1) \in \mathbb{C}[[X_1,Y_1]]$ and $F_1 = F_1(X,Y)$, $F_2 = F_2(X,Y)$ are without constant terms then the substitution $G(F_1(X,Y),F_2(X,Y))$ is well defined and has usual properties. Moreover we define $\frac{\partial}{\partial X}\sum c_{\alpha\beta}X^\alpha Y^\beta = \sum \alpha c_{\alpha\beta}X^{\alpha-1}Y^\beta$, $\frac{\partial}{\partial Y}\sum c_{\alpha\beta}X^\alpha Y^\beta = \sum \beta c_{\alpha\beta}X^\alpha Y^{\beta-1}$. If $F = \sum c_{\alpha\beta}X^\alpha Y^\beta$ then

$$c_{\alpha\beta} = \frac{1}{\alpha!\beta!}\frac{\partial^{\alpha+\beta}F}{\partial X^\alpha \partial Y^\beta}(0,0)\ .$$

Similarly, we define the formal power series in one variable X. The set of all power series of the form $\sum a_\nu X^\nu$ is denoted by $\mathbb{C}[[X]]$. Clearly $\mathbb{C}[[X]] \subset \mathbb{C}[[X,Y]]$ and any power series $F = F(X,Y) \in \mathbb{C}[[X,Y]]$ can be written in the form $F = \sum F_\nu(X)Y^\nu$ where $F_\nu(X) \in \mathbb{C}[[X]]$ for $\nu \geq 0$.

Let $r,s > 0$ and put $|F|_{r,s} = \sum |c_{\alpha\beta}r^\alpha s^\beta$ for $F = \sum c_{\alpha\beta}Y^\alpha Y^\beta$. A power series F is convergent if $|F|_{r,s} < +\infty$ for some $r,s > 0$. The set of convergent power series is denoted $\mathbb{C}\{X,Y\}$. Then $\mathbb{C}\{X,Y\}$ is a subalgebra of $\mathbb{C}[[X,Y]]$. If $G \in \mathbb{C}\{X_1,Y_1\}$ and $F_1,F_2 \in \mathbb{C}\{X,Y\}$ are without constant terms then $G(F_1,F_2) \in \mathbb{C}\{X,Y\}$. Moreover, $\frac{\partial}{\partial X}\mathbb{C}\{X,Y\} \subset \mathbb{C}\{X,Y\}$ and $\frac{\partial}{\partial Y}\mathbb{C}\{X,Y\} \subset \mathbb{C}\{X,Y\}$. A power series $U = U(X,Y) \in \mathbb{C}\{X,Y\}$ is a unit in $\mathbb{C}\{X,Y\}$ if and only if it is a unit in $\mathbb{C}[[X,Y]]$.

Let $k > 0$ be an integer. We say that $F = \sum F_\nu(X)Y^\nu \in \mathbb{C}[[X,Y]]$ is distinguished in Y (with order k) if $\operatorname{ord} F(0,Y) = k$ i.e. $F(0,Y) \sim Y^k$. A polynomial $Y^k + A_1(X)Y^{k-1} + \ldots + A_k(X)\mathbb{C}[[X]][Y]$ is called distinguished if $A_1(0) = \ldots = A_k(0) = 0$. In what follows we formulate the theorems for the case of convergent power series.

Theorem 3.1 (Späth Division Theorem)
Let $F = F(X,Y) \in \mathbf{C}\{X,Y\}$ be distinguished in Y with order $k > 0$. Then for every power series $G = G(X,Y) \in \mathbb{C}\{X,Y\}$ there exist unique $Q \in \mathbb{C}\{X,Y\}$ and unique $R \in \mathbb{C}\{X\}[Y]$ with $\deg_Y R < k$ such that

$$G = QF + R\ .$$

Proof. O. Zariski and P. Samuel, Commutative Algebra, vol. II.

Theorem 3.2 (Weierstrass' Preparation Theorem)
Let $F = F(X,Y) \in \mathbf{C}\{X,Y\}$ be distinguished in Y with order $k > 0$. Then there exists unique distinguished polynomial $W = W(X,Y) = Y^k + A_1(X)Y^{k-1} + \ldots + A_k(X)$ of degree k such that $F \sim W$.

Proof. We deduce Theorem 3.2 from Theorem 3.1. Let us apply the division theorem to $G = Y^k$ and F. Then $Y^k = QF + R$ where $R \in \mathbb{C}\{X\}[Y]$ is of degree in Y less than k. Let $W = Y^k - R$. Then $W = QF$ and $\operatorname{ord} W(0,Y) = \operatorname{ord} Q(0,Y) + \operatorname{ord} F(0,Y)$. Since $\operatorname{ord} W(0,Y) \leq k$ and $\operatorname{ord} F(0,Y) = k$ then we get $\operatorname{ord} Q(0,Y) = 0$ i.e. $Q(0,0) \neq 0$ and $\operatorname{ord} W(0,Y) = k$. Thus $F \sim W$. The uniqueness follows from the uniqueness of Q and R in the division theorem ∎

Corollary 3.3 (Implicit function theorem)
Let $F = F(X,Y) \in \mathbb{C}\{X,Y\}$, where $F(0,0) = 0$ and $\frac{\partial F}{\partial Y}(0,0) \neq 0$. Then there is exactly one series $\varphi \in \mathbb{C}\{X\}$ such that $\varphi(0) = 0$ and $F(X, \varphi(X)) = 0$.

Theorem 3.4 (Hensel's lemma)
Let $F(X,Y) = Y^n + F_1(X)Y^{n-1} + \ldots + F_n(X) \in \mathbb{C}\{X\}[Y]$ be a polynomial of degree $n > 0$ such that $F(0,Y) = (Y - c_1)^{n_1} \ldots (Y - c_s)^{n_s}$, where $c_i \neq c_j$ for $i \neq j$. Then $F(0,Y) = W_1(X, Y - c_1) \ldots W_s(X, Y - c_1)$ in $\mathbb{C}\{X\}[Y]$ where $W_j = W_j(X, Y_j)$ are distinguished polynomials of degree n_j for $j = 1, \ldots, s$.

Proof. We proceed by induction: for $s = 1$ there is nothing to show. Let $s > 1$ and consider $F(X, Y_1 + c_1) \in \mathbb{C}\{X, Y_1\}$. It is a distinguished in Y_1 power series with order $n_1 > 0$. Therefore $F(X, Y_1 + c_1) = W_1(X, Y_1)Q(X, Y_1)$ where $Q(X, Y_1)$ is a unit. We check that $Q(X, Y_1)$ is a polynomial in Y_1 and $F(X,Y) = W_1(X, Y - c_1)Q(X, Y - c_1)$. We have $Q(0, Y - c_1) = (Y - c_2')^{n_2} \ldots (Y - c_s')^{n_s}$ where $c_2' = c_2 - c_1$, …, $c_s' = c_s - c_1$. If suffices to apply the induction hypothesis to $Q(X, Y - c_1)$ ■

Corollary 3.5 *Suppose that the polynomial $F(X,Y) = Y^n + F_2(X)Y^{n-2} + \ldots + F_n(X) \in \mathbf{C}\{X\}[Y]$, $n > 1$ is not distinguihed i.e. $F(0,Y) \neq Y^n$ (note that the coefficient of Y^{n-1} is zero!). Then F is reducible i.e. it is a product of polynomials of lower degree.*

Proof. The polynomial $F(0,Y) \in \mathbb{C}[Y]$ has at least two different roots in $\mathbb{C}$ (if $F(0,Y) = (Y - c)^n$ with $c \neq 0$ then the coefficient of Y^{n-1} is $\neq 0$). Use Hensel's lemma ■

Theorem 3.6 (Puiseux' Theorem, first version)
Let $F(X,Y) = Y^n + F_1(X)Y^{n-1} + \ldots + F_n(X) \in \mathbb{C}\{X\}[Y]$. Then there exists an integer $e > 0$ and a power series $\psi(T)$ of one variable T such that $F(T^e, \psi(T)) = 0$.

Proof. Suppose that $n > 1$ and that Puiseux' theorem is true for all polynomials of degree $< n$. Substituting $\tilde{Y} = Y + \frac{1}{n}F_1(X)$ we get $\tilde{F}(X, \tilde{Y}) = \tilde{Y}^n + \tilde{F}_2(X)\tilde{Y}^{n-2} + \ldots + \tilde{F}_n(X)$. Therefore we may assume in the sequel that $F_1(X) = 0$. If $F(0,Y) \not\equiv Y^n$ then F is reducible by Corollary 3.5 and the theorem is true for F by induction hypothesis. Let us suppose that F is distinguished and $F(X,Y) \not\equiv Y^n$. Let $I = \{i \in [2,n] : F_i(X) \neq 0 \text{ in } \mathbb{C}\{X\}\}$ and write $\min_{i \in I}\{\frac{1}{i}\operatorname{ord} F_i\} = \frac{p}{q}$ with coprime integers $p, q > 0$. Let X_1, Y_1 be new variables. Inserting $X = X_1^q$ and $Y = X_1^p Y_1$ to $F(X,Y)$ we get $F(X_1^q, X_1^p Y_1) = X_1^{pn} Y_1^p + F_2(X_1^q) X_1^{p(n-2)} Y_1^{n-2} + \ldots + F_n(X_1^q) = X_1^{pm} F_1(X_1, Y_1)$ in $\mathbf{C}\{X_1, Y_1\}$ since $\operatorname{ord} F_i(X_1^a) X_1^{p(n-i)} = (\operatorname{ord} F_i)q + p(n-i) \geq (i\frac{p}{q})q + p(n-i) = pn$ for $i \geq 2$. Moreover, there is $i \geq 2$ such that $\operatorname{ord} F_i(X_1^q) X_1^{p(n-i)} = pn$. Therefore the polynomial $F_1(X_1, Y_1) = Y_1^n + \ldots$ satisfies the assumptions of Corollary 3.5 and is reducible. Applying the induction hypothesis to a factor of $F_1(X_1, Y_1)$ we get an integer $e_1 > 0$ and a power series $\psi_1(T)$ such that $F_1(T^{e_1}, \psi_1(T)) = 0$. Consequently $F(T^{e_1 q}, T^{e_1 p}\psi_1(T)) = 0$ and it suffices to take $e = e_1 q$ and $\psi(T) = T^{e_1 p}\psi_1(T)$.

4 Parametrizations

Let T be a variable. A parametrization is a pair $(\varphi(T), \psi(T))$ of convergent power series such that $\varphi(0) = \psi(0) = 0$ and $\varphi(T) \neq 0$ or $\psi(T) \neq 0$. Two parametrizations $(\varphi(T), \psi(T))$ and $(\varphi_1(T_1), \psi_1(T_1))$ are equivalent if there exists a convergent power series $\tau(T)$, $\operatorname{ord}\tau(T) = 1$ such that $\varphi(T) = \varphi_1(\tau(T))$, $\psi(T) = \psi_1(\tau(T))$. A parametrization $(\varphi(T), \psi(T))$ is good if there does not exist $\tau(T)$, $\operatorname{ord}\tau(T) > 1$ and a parametrization $(\varphi_1(T_1), \psi_1(T_1))$ such that $\varphi(T) = \varphi_1(\tau(T))$, $\psi(T) = \psi_1(\tau(T))$.

Property 4.1 *Any parametrization $(\varphi(T), \psi(T))$ with $\varphi(T) \neq 0$ is equivalent to a parametrization of the form $(T_1^n, \psi_1(T_1))$.*

Proof. Let $n = \operatorname{ord}\varphi(T)$. Write $\varphi(T) = T^n U(T)$ with $U(0) \neq 0$. Using Hensel's lemma we check that there exists a power series $V(T)$ such that $U(T) = V(T)^n$. Let $\tau(T) = T\,V(T)$. Since $\operatorname{ord}\tau(T) = 1$ we may write $\psi(T) = \psi_1(\tau(T))$ where $\psi_1(T_1)$ is a power series. The parametrization $(\varphi(T), \psi(T))$ is equivalent to the parametrization $(T_1^n, \psi_1(T_1))$ ∎

Property 4.2 *A parametrization of the form $(T^n, c_1T^{n_1} + c_2T^{n_2} + \dots)$ where $c_1, c_2, \dots \neq 0$ is good if and only if $\gcd(n, n_1, n_2, \dots) = 1$.*

We leave to the reader the proof of Property 4.2.

A parametrization of the form $(T^n, c_1T^{n_1} + c_2T^{n_2} + \dots)$, where $\gcd(n, n_1, n_2, \dots) = 1$ is called Puiseux' parametrization. The expression $c_1X^{n_1/n} + c_2X^{n_2/n} + \dots$ called Puiseux' series is frequently used instead of Puiseux parametrization. Let $U(n) = \{\varepsilon \in \mathbb{C} : \varepsilon^n = 1\}$.

Property 4.3 *Let $(T^n, \psi(T))$ be a Puiseux' parametrization. Then $\psi(\varepsilon_1 T) = \psi(\varepsilon_2 T)$ for $\varepsilon_1, \varepsilon_2 \in U(n)$ implies $\varepsilon_1 = \varepsilon_2$.*

Proof. From $\psi(\varepsilon_1 T) = \psi(\varepsilon_2 T)$ it follows $\psi(\varepsilon T) = \psi(T)$ where $\varepsilon = \varepsilon_1\varepsilon_2^{-1}$. Therefore it suffices to check that $\psi(\varepsilon T) = \psi(T)$ implies $\varepsilon = 1$. Let $\psi(T) = c_1T^{n_1} + c_2T^{n_2} + \dots$ with $c_1, c_2, \dots \neq 0$. From $\psi(\varepsilon T) = \psi(T)$ we get $\varepsilon^{n_1} = 1$, $\varepsilon^{n_2} = 1$, ... Since $\varepsilon^n = 1$ and $n, n_1, n_2, \dots$ have no common factor greater than 1, we can find an integer m and integers $a, a_1, \dots, a_m$ such that $an + a_1n_1 + \dots + a_mn_m = 1$. Consequently, we get $\varepsilon = \varepsilon^{an+a_1n_1+\dots+a_mn_m} = (\varepsilon^n)^a(\varepsilon^{n_1})^{a_1}\dots(\varepsilon^{n_m})^{a_m} = 1$ ∎

Proposition 4.4 *Let $(T^n, \psi(T))$ be a Puiseux' parametrization. Then there exists a distinguished polynomial $F(X,Y) = Y^n + F_1(X)Y^{n-1} + \dots + F_n(X)$ such that $F(T^n, Y) = \prod_{\varepsilon^n=1}(Y - \psi(\varepsilon T))$. The polynomial $F(X,Y)$ is irreducible in $\mathbb{C}\{X\}[Y]$.*

Proof. Let us begin with the following

Claim. Let $H(T)$ be a power series of one variable such that $H(\varepsilon T) = H(T)$ for all $\varepsilon \in U(n)$. Then there exists a power series $H_0(T_0)$ such that $H(T) = H_0(T^n)$.

Proof of the claim. Each series $H(T)$ of one variable can be written in the form $H(T) = H_0(T^n) + H_1(T^n)T + \dots + H_{n-1}(T^n)T^{n-1}$. Let $\varepsilon \in U(n)$ be a primitive root of unity and suppose that $H(\varepsilon T) = H(T)$. Then $H_k(T^n)\varepsilon^k = H_k(T^n)$ for $k = 0, \dots, n-1$ which implies $H_k(T^n) = 0$ for $k > 0$ since $\varepsilon^k \neq 1$ for $k = 1, 2, \dots, n-1$ ∎

To prove the proposition consider the product

$$\prod_{\varepsilon \in U(n)} (Y - \psi(\varepsilon T)) = Y^n + \tilde{F}_1(T)Y^{n-1} + \dots + \tilde{F}_n(T)\,.$$

It is easy to check that $\tilde{F}_k(\varepsilon T) = \tilde{F}_k(T)$ for all $\varepsilon \in U(n)$ and $k = 1, \dots, n$. By the claim we get $\tilde{F}_k(T) = F_k(T^n)$ and it suffices to take $F(X,Y) = Y^n + F_1(X)Y^{n-1} + \dots + F_n(X)$.

To check that $F(X,Y) \in \mathbb{C}\{X\}[Y]$ is irreducible suppose that $G(X,Y)$ divides $F(X,Y)$. Then $G(T^n, \psi(\varepsilon_1 T)) = 0$ for a root $\varepsilon_1 \in U(n)$ and substituting $\varepsilon\varepsilon_1^{-1}$ for T, $\varepsilon \in U(n)$ shows that $G(T^n, \psi(\varepsilon T)) = 0$ for $\varepsilon \in U(n)$. From Property 4.3 it follows that $\prod_{\varepsilon\in U(n)}(Y - \psi(\varepsilon T)) = F(T^n, Y)$ divides $G(T^n, Y)$ which implies that $F(X,Y)$ divides $G(X,Y)$. Thus we get $G(X,Y) = F(X,Y)$ ∎

Theorem 4.5 (Puiseux' Theorem, second version)
Suppose that $F = F(X,Y) = Y^n + F_1(X)T^{n-1} + \ldots + F_n(X)$ is a distinguished polynomial, irreducible in $\mathbb{C}\{X\}[Y]$. Then there is a Puiseux' parametrization $(T^n, \psi(T))$ such that

$$F(T^n, Y) = \prod_{\varepsilon \in U(n)} (Y - \psi(\varepsilon T)) .$$

Proof. According to the first version of Puiseux' theorem there exist an integer $e > 0$ and a power series $\psi(T)$ such that $F(T^e, \psi(T)) = 0$. Since F is distinguished we have $\psi(0) = 0$. Let $e > 0$ be the minimal integer such that $F(T^e, \psi(T)) = 0$ for a series $\psi(T)$. Then $(T^e, \psi(T))$ is a Puiseux' parametrization by Property 4.2 and from Proposition 4.4 easily follows that $e = n$ ■

Using the Weierstrass Preparation Theorem and Puiseux' Theorem we get

Theorem 4.6 (Normalization Theorem)
Let $F = F(X,Y)$ be an irreducible power series. Then there exists a good parametrization $(\varphi(T), \psi(T))$ such that $F(\varphi(T), \psi(T)) = 0$. Any two such parametrizations are equivalent.

5 Local analytic curves

Convergent power series defined functions in neighbourhoods (nbhds) of the origin, and the size of the nbhd depends on the series. Let $F \in \mathbb{C}\{X,Y\}$ be a nonzero power series without constant term, convergent in a nbhd $\mathcal{U}$ of $0 \in \mathbb{C}^2$. We put $V(F,\mathcal{U}) = \{(x,y) \in U : F(x,y) = 0\}$. Using the Weierstrass Preparation Theorem and well-known properties of polynomials we check

Lemma 5.1 *Let $\mathcal{U}$ be a sufficiently small nbhd of the origin (i.e. such that all considered power series are convergent in $\mathcal{U}$). Then*

(a) *The sets $V(F,\mathcal{U})$ and $\mathcal{U} \setminus V(F,\mathcal{U})$ are infinite.*

(b) *If F, G are relatively prime then the set $V(F,\mathcal{U}) \cap V(G,\mathcal{U})$ is finite.*

(c) *If F has no multiple factors and $V(F,\mathcal{U}) \subset V(G,\mathcal{U})$ then F divides G.*

In what follows we are interested in properties of $V(F,\mathcal{U})$ which does not depend on the nbhd $\mathcal{U}$. Observe that if F and G have no multiple factors then $V(F,\mathcal{U}) = V(G,\mathcal{U})$ for a nbhd of $0 \in \mathbb{C}^2$ if and only if $F \sim G$. The above observation leads to the following

Definition 5.2 Let $F = F(X,Y) \in \mathbb{C}\{X,Y\}$ be a nonzero power series without constant term. Then the local (analytic) curve $\{F = 0\}$ is the set

$$\{G \in \mathbb{C}\{X,Y\} : G \sim F \text{ in } \mathbb{C}\{X,Y\}\} .$$

Note that $\{F_1 = 0\} = \{F_2 = 0\}$ if and only if $F_1 \sim F_2$. The local curve $\{F = 0\}$ is called reduced (resp. irreducible) if the power series F has no multiple factors (resp. is irreducible). The local irreducible curves are also called branches. In what follows we say that the set $V(F,\mathcal{U})$ represents the local reduced curve $\{F = 0\}$ in the nbhd $\mathcal{U}$. Two local reduced curves $\{F = 0\}$ and $\{G = 0\}$ are analytically equivalent (a.e.) if there exists a pair of convergent power series $\Phi(X,Y) = (aX + bY + \ldots, cX + dY + \ldots)$ where $ad - bc \neq 0$ such that $F \circ \Phi \sim G$. This is equivalent to the following condition: there exist nbhds $\mathcal{U}, \mathcal{U}'$

of $0 \in \mathbb{C}^2$ and a bianalytic mapping $\mathcal{U} \to \mathcal{U}'$ which maps the set representing $\{F=0\}$ in $\mathcal{U}$ on the set representing $\{G=0\}$ in $\mathcal{U}'$. A local curve $\{F=0\}$ is singular of multiplicity m if $m = \operatorname{ord} F > 1$. Any nonsingular local curve (of multiplicity 1) is a.e. to the line $\{Y=0\}$; any local curve of multiplicity 2 is a.e. to the local curve $\{Y^2 - X^k = 0\}$ where $k \geq 2$. A function defined on the set of reduced local curves is an analytic invariant if it is constant on a.e. curves. For any local curves $\{F=0\}$, $\{G=0\}$ we define the intersection number $i_0(F,G)$:

Definition 5.3 Let F, G be nonzero power series without constant terms and let $F = F_1 \ldots F_m$ in $\mathbb{C}\{X,Y\}$ with irreducible factors F_i, $i = 1, \ldots, m$. Let $(\varphi_i(T_i), \psi_i(T_i))$ be a good parametrizations such that $F_i(\varphi_i(T_i), \psi_i(T_i)) = 0$ in $\mathbb{C}\{T_i\}$. Then

$$i_0(F,G) = \sum_{i=1}^{m} \operatorname{ord} G(\varphi_i(T_i), \psi_i(T_i)) \ .$$

The following properties of intersection multiplicity are basic for us.

(i) $i_0(F,G) = +\infty$ if and only if F and G have a common factor in $\mathbb{C}\{X,Y\}$,

(ii) $i_0(F, G_1G_2) = i_0(F,G_1) + i_0(F,G_2)$,

(iii) $i_0(F, G+HF) = i_0(F,G)$,

(iv) $i_0(F,G) = i_0(G,F)$,

(v) if $\Phi(X,Y) = (aX + bY + \ldots, cX + dY + \ldots)$ where $ad - bc \neq 0$ then

$$i_0(F \circ \Phi, G \circ \Phi) = i_0(F,G) \ ,$$

(vi) $i_0(F,G) = 1$ if and only if $\frac{\partial F}{\partial X}(0,0)\frac{\partial G}{\partial Y}(0,0) - \frac{\partial F}{\partial Y}(0,0)\frac{\partial G}{\partial X}(0,0) \neq 0$.

In what follows we put $i_0(F,G) = 0$ if $F(0) \neq 0$ or $G(0) \neq 0$. The reader will find the detailed proofs of the above properties in A. Płoski, Introduction to the local theory of plane algebraic curves, in Analytic and algebraic geometry (eds T. Krasiński and St. Spodzieja) Łódź University Press, Łódź 2013.

6 The Milnor number

For every nonzero power series $F \in \mathbb{C}\{X,Y\}$ without constant term we define the Milnor number

$$\mu_0(F) = i_0\left(\frac{\partial F}{\partial X}, \frac{\partial F}{\partial Y}\right) \ .$$

Note that $\mu_0(F) < +\infty$ if and only if F has no multiple factors. We will get the main properties of the Milnor number from Teissier's lemma:

Lemma 6.1 *Let $F = F(X,Y)$ be an $Y-$distinguished power series with no multiple factors. Then*

$$i_0\left(F, \frac{\partial F}{\partial Y}\right) = \mu_0(F) + i_0(F,X) - 1 \ .$$

Proof. If $\frac{\partial F}{\partial Y}(0,0) \neq 0$ then the lemma is obvious. Let $\frac{\partial F}{\partial Y} = G_1 \dots G_s$ with irreducible $G_1, \dots, G_s \in \mathbb{C}\{X,Y\}$ and let $p_i(T_i) = (\varphi_i(T_i), \psi(T_i))$ be a good parametrization of the branch $G_i(X,Y) = 0$. From $\frac{\partial F}{\partial Y}(p_i(T_i)) = 0$ in $\mathbb{C}\{T_i\}$ we get $\frac{d}{dT_i}F(p_i(T_i)) = \frac{\partial F}{\partial X}(p_i(T_i))\frac{d\varphi_i}{dT_i}$, hence $\operatorname{ord} F(p_i(T_i)) = \operatorname{ord} \frac{\partial F}{\partial X}(p_i(T_i)) + \operatorname{ord} \varphi(T_i)$ for $i = 1, \dots, s$. Using the definition of intersection multiplicity we get

$$\begin{aligned} i_0\left(F, \frac{\partial F}{\partial Y}\right) &= \sum_{i=1}^{s} i_0(F, G_i) = \sum_{i=1}^{s} \operatorname{ord} F(p_i(T_i)) \\ &= \sum_{i=1}^{s} \operatorname{ord} \frac{\partial F}{\partial X}(p_i(T_i)) + \sum_{i=1}^{s} \operatorname{ord} \varphi_i(T_i) \\ &= \sum_{i=1}^{s} i_0\left(\frac{\partial F}{\partial X}, G_i\right) + \sum_{i=1}^{s} i_0(X, G_i) = \mu_0(F) + i_0\left(X, \frac{\partial F}{\partial Y}\right) \\ &= \mu_0(F) + i_0(F, X) - 1 \ . \end{aligned}$$

Property 6.2

(a) *If* $\Phi(X,Y) = (aX + bY + \dots, cX + dY + \dots)$ *where* $ad - bc \neq 0$ *then* $\mu_0(F) = \mu_0(F \circ \Phi)$.

(b) *If* $G \sim F$ *then* $\mu_0(G) = \mu_0(F)$.

Proof. The proof of (a) we leave to the reader. To check (b) assume that F is $Y-$distinguished. We have $i_0(F, \partial F/\partial Y) = i_0(G, \partial G/\partial Y)$ and $i_0(F,X) = i_0(G,X)$ by properties of intersection numbers. Therefore (b) follows from Teissier's lemma ■

Property 6.3 *If* $F = F_1 \dots F_m$ *is a product of pairwise coprime* F_i *then*

$$\mu_0(F) + m - 1 = \sum_{i=1}^{m} \mu_0(F_i) + 2 \sum_{1 \leq i < j \leq m} i_0(F_i, F_j) \ .$$

Proof. By basic properties of intersection numbers we get

$$i_0\left(F, \frac{\partial F}{\partial Y}\right) = \sum_{i=1}^{m} i_0\left(F_i, \frac{\partial F_i}{\partial Y}\right) + 2 \sum_{1 \leq i < j \leq m} i_0(F_i, F_j) \ .$$

To obtain the formula it suffices to apply Teissier's lemma to power series $F, F_1, \dots, F_m$ ■

Let $F = F(X,Y) \in \mathbb{C}\{X,Y\}$ be an irreducible power series of order $n > 1$. We may assume that $F = Y^n +$ terms of order greater than n. Let Y_1 be a new variable. A power series $F_1 = F_1(X, Y_1)$ is a proper transform of $F(X,Y)$ (by the quadratic transformation $Y = XY_1, X = X$) if $F(X, XY_1) = X^n F_1(X, Y_1)$. Note that $\operatorname{ord} F_1(0, Y_1) = \operatorname{ord} F(0,Y) = n$.

Property 6.4 *If* F *is an irreducible power series,* F_1 *its proper transform then* $\mu_0(F) = (\operatorname{ord} F)(\operatorname{ord} F - 1) + \mu_0(F_1)$.

Proof. If $p(T) = (T^n, \psi(T))$ is a Puiseux' parametrization of $F(X,Y) = 0$ then $\operatorname{ord} \psi(T) > n$ and $p_1(T) = (T^n, \psi(T)/T^n)$ is a Puiseux' parametrization of $F_1(X, Y_1) = 0$. A simple computation shows that $i_0(F, \partial F/\partial Y) = i_0(F_1, \partial F_1/\partial Y_1) + n(n-1)$, hence $\mu_0(F) = n(n-1) + \mu_0(F_1)$ by Teissier's lemma ■

By repeated application of Property 6.4 we get

Property 6.5 *For every irreducible power series $F \in \mathbb{C}\{X,Y\}$ there exist a sequence of irreducible series $F_0,\dots,F_s$ such that $F_0 = F$, $F_{i+1} = 0$ is a proper transformation of $F_i = 0$ and $F_s = 0$ is nonsingular. Moreover, $\mu_0(F) = \sum_{i=1}^{s}(\operatorname{ord} F_i)(\operatorname{ord} F_i - 1)$.*

Note that the above formula for $\mu_0(F)$ implies that the Milnor number of a branch is always an even number. Now, we may formulate the main result of this section.

Theorem 6.6 *There is a unique analytic invariant δ_0 defined on reduced algebroid curves such that*

(i) *if* $\operatorname{ord} F = 1$ *then $\delta_0(F) = 0$,*

(ii) *if $F = 0$ is an irreducible, singular curve then $\delta_0(F) = \frac{1}{2}(\operatorname{ord} F)(\operatorname{ord} F - 1) + \delta_0(F_1)$ where $F_1 = 0$ is a proper transformation of $F = 0$,*

(iii) *if $F = F_1 \dots F_r$ is a product of pairwise coprime power series F_i ($r = r(F)$) then*

$$\delta_0(F) = \sum_{i=1}^{r} \delta_0(F_i) + \sum_{1 \le i < j \le r} i_0(F_i, F_j)\ .$$

Moreover, the Milnor formula $\mu_0(F) = 2\,\delta_0(F) - r(F) + 1$ holds for any reduced local curve $F = 0$.

Proof. The uniqueness of δ_0 follows immediately from Property 6.5 by induction on the numbers of quadratic transformations needed to desingularize a branch. We prove simultaneously the existence of δ_0 and the Milnor formula by putting $\delta_0(F) = \frac{1}{2}(\mu_0(F) + r(F) - 1)$ and by using the Properties 6.2, 6.3 and 6.4 ∎

Example 6.7 The reduced curve $F = 0$ has an ordinary r − fold singularity if it has r branches, all non-singular and intersecting each other with multiplicity 1. For such a curve $\mu_0 = (r-1)^2$ and $\delta_0 = \frac{1}{2}r(r-1)$.

For an elementary, intersection–theoretical approach to the local invariants of plane curve singularities we refer the reader to Pi. Cassou-Nogues, A. Płoski, Invariants of plane curve singularities and Newton diagrams, Univ. Iag. Acta Math. Fasc. XLIX, 2011, pp. 9–34.

7 Puiseux characteristic

Let $\{F(X,Y) = 0\}$ be an irreducible singular local curve. Assume that $\operatorname{ord} F(0,Y) = \operatorname{ord} F$ (if F is irreducible then $\operatorname{ord} F(X,0) = \operatorname{ord} F$ or $\operatorname{ord} F(0,Y) = \operatorname{ord} F$) and let $(T^n, \psi(T)) = (T^n, c_1T^{n_1} + c_2T^{n_2} + \dots)$ where $c_1, c_2, \dots \neq 0$ be a Puiseux' parametrization such that $F(T^n, \psi(T))$. Then $n = \operatorname{ord} F > 1$ and $n \le n_1 < n_2 < \dots$. Recall that $\gcd(n, n_1, n_2, \dots) = 1$. A characteristic exponent is, by definition, each exponent n_k such that $\gcd(n, n_1, \dots, n_{k-1}) \neq \gcd(n, n_1, \dots, n_k)$. The set of characteristic exponents is finite; if $\beta_1 < \dots < \beta_g$ are all characteristic exponents then we put $\beta_0 = n$ and call the sequence $(\beta_0, \beta_1, \dots, \beta_g)$ the characteristic of the branch $\{F = 0\}$.

Let $e_k = \gcd(\beta_0, \dots, \beta_k)$ for $k = 0, 1, \dots, g$. Then $n = e_0 > e_1 > \dots > e_{g-1} > e_g = 1$ is a sequence of divisors of the number $n = \operatorname{ord} F$. Let $U(n) = \{\varepsilon \in \mathbf{C} : \varepsilon^n = 1\}$. Then $U(n) = U(e_0) \supset U(e_1) \supset \dots \supset U(e_g) = \{1\}$ and $U(e_{k-1}) \neq U(e_k)$ for $k = 1, \dots, g$.

Property 7.1 *If $\varepsilon \in U(e_{k-1}) \setminus U(e_k)$ then $\operatorname{ord}(\psi(\varepsilon T) - \psi(T)) = \beta_k$.*

Proof. Fix $k \in \{1, \dots, g\}$ and write $\psi(T) = \psi_k(T^{e_{k-1}} + cT^{\beta_k} + \dots$ (terms of order $> \beta_k$) where $c \neq 0$. Then $\psi(\varepsilon T) - \psi(T) = c(\varepsilon^{\beta_k} - 1)T^{\beta_k} + \dots$ and $\operatorname{ord}(\psi(\varepsilon T) - \psi(T)) = \beta_k$ since $\varepsilon^{\beta_k} \neq 1$ ■

Proposition 7.2 *Let $\{F(X,Y) = 0\}$ be a singular branch with characteristic $(\beta_0, \beta_1, \dots, \beta_g)$. Then*

$$\mu_0(F) = \sum_{k=1}^{g} (e_{k-1} - e_k)\beta_k - \beta_0 + 1 \ .$$

Proof. Assume that $\operatorname{ord} F(0,Y) = \operatorname{ord} F$ and let $(T^n, \psi(T))$ be a Puiseux' parametrization such that $F(T^n, \psi(T)) = 0$. A simple computation shows that

$$\frac{\partial F}{\partial Y} = \prod_{\varepsilon \neq 1} (Y - \psi(\varepsilon T))U(T,Y) + (Y - \psi(T))G(T,Y)$$

in $\mathbb{C}\{T,Y\}$ where $U(0,0) \neq 0$. Hence we get

$$\begin{aligned} i_0\left(F, \frac{\partial F}{\partial Y}\right) &= \operatorname{ord} \frac{\partial F}{\partial Y}(T^n, \psi(T)) = \sum_{\varepsilon \neq 1} \operatorname{ord}(\psi(\varepsilon T) - \psi(T)) \\ &= \sum_{k=1}^{g} \#(U(e_{k-1}) \setminus U(e_k))\beta_k = \sum_{k=1}^{g} (e_{k-1} - e_k)\beta_k \end{aligned}$$

by Property 7.1. By Teissier's lemma we get

$$\mu_0(F) = i_0\left(F, \frac{\partial F}{\partial Y}\right) - i_0(F, X) + 1 = \sum_{k=1}^{g} (e_{k-1} - e_k)\beta_k - \beta_0 + 1 \qquad ■$$

Remark 7.3 If $\{F = 0\}$ is a branch with characteristic $(\beta_0, \beta_1, \dots, \beta_g)$ then we put $m_k = \beta_k/e_k$, $n_k = e_{k-1}/e_k$ for $k = 1, \dots, g$ and call $(m_1, n_1), \dots, (m_g, n_g)$ the characteristic pairs of the branch.

Two reduced local curves $\{F = 0\}$ and $\{G = 0\}$ have the same topological type if there exist nbhds $\mathcal{U}, \mathcal{U}'$ of $0 \in \mathbb{C}^2$ and a homomorphisms $\mathcal{U} \to \mathcal{U}'$ which maps the set representing $\{F = 0\}$ in $\mathcal{U}$ on the set representing $\{G = 0\}$ in $\mathcal{U}'$. In 1929, K. Brauner proved the following theorem.

Theorem 7.4 *Let $\{F = 0\}$ be a singular branch with characteristic $(\beta_0, \dots, \beta_g)$. Then it has the same topological type as the branch defined by Puiseux' parametrization*

$$\begin{aligned} X &= T^n \ , \\ Y &= T^{\beta_1} + T^{\beta_2} + \dots + T^{\beta_g} \ . \end{aligned}$$

In 1932, W. Burau and O. Zariski proved that the converse of the above theorem is true.

Theorem 7.5 *Let $\{F = 0\}$ be a singular branch. Then the Puiseux characteristic $(\beta_0, \dots, \beta_g)$ is an invariant of topological type of $\{F = 0\}$.*

Finally, M. Lejeune-Jalabert and O. Zariski proved the following theorem

Theorem 7.6 *Let $\{F = 0\}$ be a reduced local analytic curve. Then the topological type of $\{F = 0\}$ is determined by the topological type of every irreducible component of $\{F = 0\}$ and all the pairs of intersection multiplicity of these components.*

From the properties of the Milnor number proved above (Property 6.3 and Proposition 7.2) and Theorem 7.6 we get

Theorem 7.7 *The Milnor number of a reduced local analytic curve is an invariant of topological type.*

Let $F(T, X, Y)$ be a convergent power series in three variables T, X, Y such that $F(T, 0, 0) = 0$ and for $t \in \mathbb{C}$ close to the origin the power series $F_t = F(t, X, Y)$ have no multiple factors. In 1976 Le Dung Trang and Ramanujam proved

Theorem 7.8 *If $\mu_0(F_t) \equiv const$ for $t \in \mathbb{C}$ close to $0 \in \mathbb{C}$ then the local analytic curves $\{F_t = 0\}$ have the same topological type.*

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