PL ISSN 0459-6854

BULLETIN

DE LA SOCIÉTÉ DES SCIENCES ET DES LETTRES DE ŁÓDŹ

1993 Vol. XLIII, 6

Recherches sur les déformations Vol. XV, 6

pp.53-57

Arkadiusz Płoski

THE NOETHER EXPONENT AND JACOBI FORMULA

Abstract

For any polynomial mapping $F = (F_1, \ldots, F_n)$ of $\mathcal{C}^n$ with a finite number of zeros we define the Noether exponent $\nu(F)$. We prove the Jacobi formula for all polynomials of degree strictly less than $\sum_{i=1}^{n}(\deg F_i - 1) - \nu(F)$.

1. The Noether exponent

If $P = P(Z)$ is a complex polynomial in n variables $Z = (Z_1, \ldots, Z_n)$ then we denote by $\tilde{P} = \tilde{P}(\tilde{Z})$, $\tilde{Z} = (Z_0, Z)$ the homogenization of P. If $\mathcal{H}$ is a set of homogeneous polynomials in $n+1$ variables then we denote by $V(\mathcal{H})$ the subset of the complex projective space $\mathcal{P}^n$ defined by equations $H = 0$, $H \in \mathcal{H}$.

The polynomial mapping $F = (F_1, \ldots, F_n) : \mathcal{C}^n \to \mathcal{C}^n$ *has a finite number of zeros* if the set $V(\tilde{F}_1, \ldots, \tilde{F}_n)$ is finite. We put $V_\infty(F) = V(\tilde{F}_1, \ldots, \tilde{F}_n, Z_0)$ and call $V_\infty(F)$ the set of zeros of F at infinity. We identify $\mathcal{C}^n$ and $\mathcal{P}^n \setminus V(Z_0)$. Clearly F has a finite number of zeros if and only if the sets $F^{-1}(0) \subset \mathcal{C}^n$ and $V_\infty(F) \subset \mathcal{P}^n$ are both finite.

Definition (1.1). Let $F = (F_1, \ldots, F_n)$ be a polynomial mapping of $\mathcal{C}^n$ with a finite number of zeros. By the Noether exponent of F we mean the smallest integer $\nu \geq 0$ such that the homogeneous forms $Z_0^\nu, \tilde{F}_1, \ldots, \tilde{F}_n$ satisfy Noether's condition at every point of the set $V_\infty(F)$ (cf. Appendix).

If $V_\infty(F) = \emptyset$ then $\nu(F) = 0$. If $V_\infty(F) \neq \emptyset$ and the hypersurfaces meet transversally at any point of $V_\infty(F)$, then $\nu(F) = 1$. For any polynomial mapping $F = (F_1, \ldots, F_n)$ with a finite number of zeros we put $\mu(F) = \sum_{z \in F^{-1}(0)} \operatorname{mult}_z F$ where $\operatorname{mult}_z F$ stands for the multiplicity of F at z. If $F^{-1}(0) = \emptyset$ then $\mu(F) = 0$.

Let $d_i = \deg F_i$ for $i = 1, \ldots, n$.

Proposition (1.2). *If F has a finite number of zeros, then* $\nu(F) \leq \prod_{i=1}^{n} d_i - \mu(F)$.

Proof. We have $\nu(F) \leq \max\{ (\tilde{F}_1, \dots, \tilde{F}_n)_p : p \in V_\infty(F)\}$ (cf. Appendix (A5)). On the other hand, by Bezout's theorem $\sum_{p \in V_\infty(F)} i(\tilde{F}_1, \dots, \tilde{F}_n)_p = \prod_{i=1}^{n} d_i - \mu(F)$ and (1.2) follows.

Remark(1.3). Let $k = \sharp V_\infty(F)$. Then a reasoning similar to the above shows that $\nu(F) \leq \prod_{i=1}^n d_i - \mu(F) - k + 1$.

Proposition (1.4). *Suppose that the polynomial mapping $F = (F_1, \dots, F_n)$ has a finite number of zeros and let P be a polynomial belonging to the ideal generated by $F_1, \dots, F_n$ in the ring of polynomials. Then there exist polynomials $A_1, \dots, A_n$ such that $P = A_1F_1 + \dots + A_nF_n$ with $\deg A_iF_i \leq \deg P + \nu(F)$ for $i = 1, \dots, n$.*

Proof. The homogeneous forms $Z_0^\nu \tilde{P}$, $\tilde{F}_1$, ..., $\tilde{F}_n$ ($\nu = \nu(F)$) satisfy Noether's conditions at every point of $V(\tilde{F}_1, \dots, \tilde{F}_n)$, then by Noether's Fundamental Theorem (cf. Appendix) there are homogeneous forms $\tilde{A}_1, \dots, \tilde{A}_n$ such that $Z_0^\nu \tilde{P} = \tilde{A}_1\tilde{F}_1 + \dots + \tilde{A}_n\tilde{F}_n$, $\deg(\tilde{A}_i\tilde{F}_i) = \deg(Z_0^\nu\tilde{P}) = \nu + \deg P$. We get (1.4) by putting $Z_0 = 1$.

For any $z = (z_1, \dots, z_n) \in C^n$ we put $|z| = \max(|z_1|, \dots, |z_n|)$. Recall that if $P : C^n \to C$ is a polynomial of degree d then there exist a constant $C > 0$ such that $|P(z)| \leq C|z|^d$ for $|z| \geq 1$.

Proposition (1.5). *Let $F = (F_1, \dots, F_n)$ be a polynomial mapping with a finite number of zeros. Then there exist positive constants C and R such that*

$$|F(z)| \geq C|z|^{\min(d_i) - \nu(F)} \quad \text{for } |z| \geq R.$$

Proof. Since the fiber $F^{-1}(0)$ is finite then there are polynomials $P_i(z_i) \not\equiv 0$ ($i = 1, \dots, n$) which belong to the ideal generated by $F_1, \dots, F_n$ in the ring of polynomials (cf. [4, p. 23]). Let $m_i = \deg P_i(z_i)$. By (1.4) we can write $P_i(z_i) = A_{i1}F_1 + \dots + A_{in}F_n$, $\deg(A_{ij}F_j) \leq m_i + \nu(F)$. Hence there exist constants $C > 0$ and $R \geq 1$ such that for every $i = 1, \dots, n$

$$|z_i|^{m_i} \leq C|z|^{m_i + \nu(F) - \min(d_i)}|F(z)|$$

if $|z_i| \geq R$, and the proposition follows.

Corollary (1.6). If $\nu(F) < \min_{i=1}^n(d_i)$ then F is proper i.e. $\lim_{|z| \to \infty} |F(z)| = +\infty$.

To end with let us note two corollaries of propositions (1.2), (1.4) and (1.5).

Corollary (1.7). Let $F = (F_1, \dots, F_n)$ be a polynomial mapping with a finite number of zeros. Let $\mu = \mu(F)$. Then

(1.7.1) (cf. [3], [10]) there is a constant $C > 0$ such that $|F(z)| \geq C|z|^{\mu - \prod d_i + \min(d_i)}$ for large $|z|$.

(1.7.2) (cf. [11]) If P belongs to the ideal generated by $F_1, \dots, F_n$ in the ring of polynomials, then $P = A_1F_1 + \dots + A_nF_n$ with $\deg(A_iF_i) \leq \prod_{i=1}^n d_i - \mu + \deg P$ for $i = 1, \dots, n$.

2. The Jacobi formula

Let $F = (F_1, \dots, F_n)$ be a polynomial mapping such that the fiber $F^{-1}(0)$ is finite and let $G : C^n \to C$ be a polynomial. We denote by $\mathrm{res}_{F,z}(G)$ the residue of the meromorphic differential form $\dfrac{G(z)}{F_1(z) \dots F_n(z)}[dZ]$, $[dZ] = dZ_1 \wedge \dots \wedge dZ_n$, at z.

The definition and all properties of residues we need are given in [5]. Let us recall that if the Jacobian $J_F = \det(\frac{\partial F_i}{\partial Z_j})$ is different from zero at $z \in F^{-1}(0)$, then $\mathrm{res}_{F,z}(G) = \dfrac{G(z)}{J_F(z)}$. The main result of this note is

Theorem (2.1). *Suppose that the polynomial mapping $F = (F_1, \dots, F_n) : \mathcal{C}^n \to \mathcal{C}^n$ has a finite number of zeros. Then the Jacobi formula*

$$\text{(J)} \qquad \sum_{z \in F^{-1}(0)} \mathrm{res}_{F,z}(G) = 0$$

is satisfied for all polynomials $G : \mathcal{C}^n \to \mathcal{C}$ of degree strictly less than $\sum_{i=1}^{n}(d_i - 1) - \nu(F)$.

Before giving the proof of (2.1) let us make some remarks. If F has no zeros at infinity i.e. if $V_\infty(F) = \emptyset$ then $\nu(F) = 0$ and (2.1) is reduced to the Griffiths–Jacobi theorem (cf. [5]). In [1], [2], [6] and [7] there are given another generalizations of the Jacobi theorem. However, these results does not imply ours. If $\nu(F) \geq \sum_{i=1}^{n}(d_i - 1)$ then the unique polynomial satisfying the assumption of (2.1) is $G \equiv 0$.

Proof of (2.1). Let Ω be the meromorphic form in $\mathcal{P}^n$ given in $\mathcal{C}^n$ by formula

$$\Omega = \frac{G(z)}{F_1(z) \dots F_n(z)}[dZ].$$

By Residue Theorem for $\mathcal{P}^n$ we get

$$\sum_{z \in F^{-1}(0)} \mathrm{res}_{F,z}(G) = - \sum_{p \in V_\infty(F)} \mathrm{res}_p \Omega.$$

It suffices to show, that $\mathrm{res}_p \Omega = 0$ for all $p \in V_\infty(F)$.

Let $W = (W_1, \dots, W_n)$ be an affine system of coordinates in an affine neighbourhood of p such that $W_1 = 0$ is the hyperplane at infinity and p has coordinates $(0, \dots, 0)$. Without loss of generality we may assume that $Z_1 = \frac{1}{W_1}$, $Z_2 = \frac{W_2 + c_2}{W_1}, \dots, Z_n = \frac{W_n + c_n}{W_1}$.

Let $P^*(W) = W_1^d P(\frac{1}{W_1}, \frac{W_2+c_2}{W_1}, \dots, \frac{W_n+c_n}{W_1})$ for any polynomial $P(Z)$ of degree d. A simple calculation shows that near $p \in \mathcal{P}^n$:

$$\Omega = \frac{-W_1^\nu G^*(W)}{F_1^*(W) \dots F_n^*(W)}[dW], \quad \nu = \sum_{i=1}^{n}(d_i - 1) - 1 - \deg G.$$

By assumptions $\nu \geq \nu(F)$, therefore $W_1^\nu G^*(W)$ belongs to the local ideal generated by $F_1^*(W), \dots, F_n^*(W)$. Consequently

$$\mathrm{res}_p \Omega = \mathrm{res}_0 \left(\frac{-W_1^\nu G^*(W)}{F_1^*(W) \dots F_n^*(W)}[dW] \right) = 0$$

and we are done.

Corollary (2.1). *If $F^{-1}(0) \neq \emptyset$ then $\nu(F) \geq \sum_{i=1}^{n}(d_i - 1) - \deg J_F$.*

Proof. If $F^{-1}(0) \neq \emptyset$ then $\sum\limits_{z \in F^{-1}(0)} \operatorname{res}_{F,z}(J_F) = \mu(F) \neq 0$, consequently we cannot have $\deg J_F < \sum_{i=1}^{n}(d_i - 1) - \nu(F)$.

Corollary (2.2). *(cf. [2]) If the hypersurfaces $\tilde{F}_i = 0$ $(1 \leq i \leq n)$ meet transversally at infinity then (J) holds for any polynomial of degree strictly less than $\sum_{i=1}^{n}(d_i - 1) - 1$.*

Proof. If $\tilde{F}_i = 0$ $(1 \leq i \leq n)$ meet transversally then $\nu(F) \leq 1$ and (2.2) follows immediately from (2.1).

Corollary (2.3). *Suppose that for any $p \in V_\infty(F)$:*
(i) the hypersurfaces $\tilde{F}_i = 0$ $(1 \leq i \leq n)$ have distinct tangent cones at p,
(ii) $\operatorname{ord}_p \tilde{F}_i \neq d_i$ $\quad (1 \leq i \leq n)$.
Then (J) holds for any polynomial of degree $\leq n - 2$.

Proof. By (A6) we have $\nu(F) \leq \max\{\sum\limits_{i=1}^{n}(\operatorname{ord}_p \tilde{F}_i - 1) + 1 : p \in V_\infty(F)\} \leq \sum\limits_{i=1}^{n}(d_i - 2) + 1$ because $\operatorname{ord}_p(\tilde{F}_i) \leq d_i - 1$ by (ii). Consequently $\sum_{i=1}^{n}(d_i - 1) - \nu(F) \leq n - 1$ and it suffices to use (2.1).

Example. Let $F(Z_1, Z_2) = (Z_1^{d_1} - 1, Z_1 Z_2 + Z_2^{d_2})$ $\quad (d_1 \geq 1,\, d_2 \geq 2)$. Then condition (i) is satisfied but (ii) fails. We have $\sum\limits_{z \in F^{-1}(0)} \operatorname{res}_{F,z}(1) = -1$, hence condition (ii) is essential.

Appendix: Noether's Conditions

Let $H_0, H_1, \ldots, H_n$ be homogeneous forms of $n + 1$ variables such that the set $V = V(H_1, \ldots, H_n)$ is finite. We denote by $\mathcal{O}_p$ the ring of holomorfic germs at $p \in \mathcal{P}^n$. Let $d_i = \deg H_i$ for $0 \leq i \leq n$.

Max Noether's Fundamental Theorem. *The following two conditions are equivalent:*
(A1) There is an equation $H_0 = A_1 H_1 + \ldots A_n H_n$ (with A_i forms of degree $d_0 - d_i$).
(A2) For any $p \in V$ there is an linear form L such that $V \cap V(L) = \emptyset$ and $\dfrac{H_0}{L^{d_0}} \in \left(\dfrac{H_1}{L^{d_1}}, \ldots, \dfrac{H_n}{L^{d_n}}\right) \mathcal{O}_p$.

The proof of Noether's theorem follows easily (cf. [4, p. 120]) from the affine version of the theorem (cf. [12]) and from the following

(A3) Property. *If H is a homogeneous form of $n+1$ variables such that $V \cap V(H) = \emptyset$, then H is not a zero-divisor modulo ideal generated by $H_1, \ldots, H_n$ in the ring of polynomials.*

Proof of (A3). If $V \cap V(H) = \emptyset$ then $H_1, \ldots, H_n, H$ form the sequence of parameters in the local ring $\mathcal{O}$ of holomorphic functions at $0 \in \mathcal{C}^{n+1}$, consequently H is not a zero-divisor $\mod (H_1, \ldots, H_n)\mathcal{O}$. Whence follows easily (A3).

We say that the sequence $H_0, H_1, \ldots, H_n$ satisfies Noether's conditions at $p \in V$ if (A2) holds true. Let $(H_1, \ldots, H_n)_p$ denotes the intersection number of $H_1, \ldots, H_n$ at p and let $\operatorname{ord}_p H$ be the order of H at p. We have the following

Criteria for Noether's conditions. *The sequence $H_0, \dots, H_n$ satisfies Noether's conditions at $p \in V$ if any of the following are true:*
(A4) $H_1, \dots, H_n$ *meet transversally at* p *and* $p \in V(H_0)$,
(A5) $\mathrm{ord}_p H_0 \geq (H_1, \dots, H_n)_p$,
(A6) $H_1, \dots, H_n$ *have distinct tangent cones at* p *and*

$$\mathrm{ord}_p H_0 \geq \sum_{i=1}^{n}(\mathrm{ord}_p H_i - 1) + 1.$$

Proof. (A5) follows from the Mutiplicity theorem (cf. [8, p. 258]), (A6) is proved in [9] (Theorem 2.3), (A4) is a special case both of (A5) and (A6).

References

[1] A. Berenstein, A. Yger, *Une formule de Jacobi et ses conséquences*, Ann. Sci. École Norm. Sup.(4) **24** (1991), 363–377.
[2] G. Biernat, *On the Jacobi–Kronecker formula for a polynomial mappping having zeros at infinity*, Bull. Soc. Sci. Lett. Łódź, (to appear).
[3] J. Chądzyński, *On proper polynomial mappings*, Bull. Polish Acad. Sci. Math. **31** (1983), 115–120.
[4] W. Fulton, *Algebraic Curves (An Introduction to Algebraic Geometry)*, New York–Amsterdam, 1969.
[5] Ph. Griffiths, *Variations on a theorem of Abel*, Invent. Math. **35** (1976), 321–390.
[6] A. G. Khovanski, *Newton polyhedra and the Euler–Jacobi formula*, Uspekhi Mat. Nauk **33,6** (1978), 245–246.
[7] M. Kreuzer, E. Kunz, *Traces in strict Frobenius algebras and strict complete intersections*, J. Reine. Angew. Math. **381** (1987), 181–204.
[8] S. Łojasiewicz, *Introduction to Complex Algebraic Geometry*, Birkhäuser Verlag, 1991.
[9] A. Płoski, *On the Noether exponent*, Bull. Soc. Sci. Lett. Łódź **XL (2), 72** (1990), 23–29.
[10] ———, *An inequality for polynomial mappings*, Bull. Polish Acad. Sci. Math., (to appear).
[11] B. Schiffman, *Degree Bounds for the Division Problem in Polynomial Ideals*, Michigan Math. J. **36** (1989), 163–171.
[12] P. Tworzewski, *A Note on the Max Noether Theorem*, Bull. Soc. Sci. Lett. Łódź **7** (1989), 1–3.

Department of Mathematics
Technical University
Al. 1000 LPP 7,
25–314 Kielce, Poland

Presented by Jacek Chądzyński at the Session of the Mathematical-Physical Commission of the Łódź Society of Sciences and Arts on June 14, 1993

WYKŁADNIK NOETHERA I FORMUŁA JACOBIEGO

S t r e s z c z e n i e

Dla każdego odwzorowania wielomianowego $F = (F_1, \dots, F_n)$ przestrzeni C^n o skończonej liczbie zer definiujemy wykładnik Noethera $\nu(F)$ a następnie dowodzimy formuły Jacobiego dla wielomianów stopnia mniejszego od $\sum_{i=1}^{n}(\deg F_i - 1) - \nu(F)$.