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Dedicated to Professor Tadeusz Tietz on his 70th birthday

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ON THE NOETHER EXPONENT

Summary

Let $F=(f_1,\dots,f_n)$ be a germ of a holomorphic mapping having an isolated zero at $0\in\mathbb{C}^n$. The Noether exponent $\rho_0(F)$ of F is the smallest integer $\rho>0$ such that every germ of order greater or equal to ρ belongs to the ideal generated by $f_1,\dots,f_n$. The aim of this paper is to give the estimation of $\rho_0(F)$ dependent on the orders $\operatorname{ord} f_i$.

1. Introduction

Let $\mathbb{C}\{z\}$ be the ring of convergent power series in variables $z=(z_1,\dots,z_n)$. If $h\in\mathbb{C}\{z\}$ and $h=\Sigma_{i=m}^{+\infty} h_i$, where h_i is a homogeneous polynomial of degree i and $h_m\neq 0$, then we put $\operatorname{ord} h=m$, $\operatorname{in} h=h_m$. We also let $\operatorname{ord} 0=+\infty$, $\operatorname{in} 0=0$. The initial form $\operatorname{in} h$ of the power series h determines an algebraic subset of the projective space $\mathbb{CP}^{n-1}$ which we denote by $V(\operatorname{in} h)$. Any sequence of n power series $f_1,\dots,f_n\in\mathbb{C}\{z\}$ without constant term induces the germ of a holomorphic mapping $F=(f_1,\dots,f_n):(\mathbb{C}^n,0)\to(\mathbb{C}^n,0)$. The local ring of F is the quotient ring $\mathbb{C}\{z\}/I(F)$, where $I(F)$ is the ideal of $\mathbb{C}\{z\}$ generated by $f_1,\dots,f_n$. This ring will be denoted by $Q_o(F)$. We say that F is finite if $\dim_{\mathbb{C}} Q_o(F) < +\infty$ and call $\mu_o(F) = \dim_{\mathbb{C}} Q_o(F)$ the multiplicity of F (see [5] for an equivalent definition). The germ F is finite if and only if F has an isolated zero. Let us recall the following well-known property of multiplicity:

THEOREM 1.1. *For any finite germ* $F = (f_1,\dots,f_n) : (\mathbb{C}^n,0) \to (\mathbb{C}^n,0)$ *we have* $\mu_o(F) \geq \operatorname{ord} f_1 \dots \operatorname{ord} f_n$. *The equality holds if and only if* $\bigcap_{i=1}^n V(\operatorname{in} f_i) = \emptyset$.

Simple proofs of (1.1) were given in [6] and [7]. In the sequel we shall also need the version of Nullstelensatz proved in [7].

THEOREM 1.2 (cf. [7]). *Let* $F = (f_1,\dots,f_n)$ *be a finite germ of multiplicity* μ. *Then for any power series* $h \in \mathbb{C}\{z\}$ *without constant term there exist series* $a_1,\dots,a_n$ *such that* $h^\mu = a_1 f_1 + \dots + a_n f_n$ *with* $\operatorname{ord}(a_i f_i) \geq \operatorname{ord} f_1 \dots \operatorname{ord} f_n \operatorname{ord} h$ *for* $i = 1,\dots,n$.

We denote by $\operatorname{Res}_F(h)$ the residue of the meromorphic differential form $h(z)/f_1(z)\dots f_n(z)dz_1 \wedge \dots \wedge dz_n$. The definition and all the properties of residues we need are given in [4]. Recall that $\operatorname{Res}_F(h) = 0$ for all $h \in I(F)$. On the other hand, $\operatorname{Res}_F(\operatorname{Jac} F) = \mu_o(F)$, where $\operatorname{Jac} F = \det(\partial f_i/\partial z_j)$, so we have

THEOREM 1.3. *If the germ* F *is finite, then the Jacobian* $\operatorname{Jac} F$ *does not belong to the ideal* $I(F)$.

2. The Noether exponent

Let $F = (f_1,\dots,f_n)$ be a finite germ. By the Noether exponent $\rho_o(F)$ of F we mean the smallest positive integer ρ such that every power series of order greater or equal to ρ belongs to the ideal $I(F)$. The Noether exponent, like the multiplicity $\mu_o(F)$, depends only on the local ring $Q_o(F)$. Indeed, if $m(Q_o(F))$ denotes the unique maximal ideal of $Q_o(F)$, then $\rho_o(F)$ equals the smallest ρ such that $m(Q_o(F))^\rho = \{0\}$. Let us note the following two lemmas:

LEMMA 2.1. *For any finite germ* $F : \rho_o(F) \geq \operatorname{ord}(\operatorname{Jac} F) + 1$.

P r o o f. Lemma 2.1 follows immediately from Theorem 1.3 and the definition of $\rho_o(F)$.

LEMMA 2.2. Let $G = (g_1, \dots, g_n)$ be a finite germ such that $g_j = a_{j1}f_1 + \dots + a_{jn}f_n$ with $a_{jk} \in \mathbb{C}\{z\}$. Then $\rho_o(F) \leq \rho_o(G) - \operatorname{ord}(\det(a_{jk}))$.

P r o o f. Let $\operatorname{ord} h \geq \rho_o(G) - \operatorname{ord} a$, where $a = \det(a_{jk})$. Then $\operatorname{ord}(hka) \geq \rho_o(G)$ for any $k \in \mathbb{C}\{z\}$ and, by the definition of $\rho_o(G)$, $hka \in I(G)$, hence $\operatorname{Res}_G(hka) = 0$ for any $k \in \mathbb{C}\{z\}$. From the transformation formula (cf. [4], chapt. 5) we get $\operatorname{Res}_F(hk) = \operatorname{Res}_G(hka) = 0$ for any $k \in \mathbb{C}\{z\}$, so the local duality theorem (cf. [4]) implies that $h \in I(F)$. Thus we have shown that the ideal $I(F)$ contains all power series of order greater than $\rho_o(G) - \operatorname{ord} a$, and therefore $\rho_o(F) \leq \rho_o(G) - \operatorname{ord} a$.

The theorem given below was proved by Chądzyński and Krasiński in the case $n = 2$ (cf. [3]).

THEOREM 2.3. Let $F = (f_1, \dots, f_n)$ be a finite germ. Then $\rho_o(F) \geq \Sigma_{i=1}^n(\operatorname{ord} f_i - 1) + 1$. The equality holds if and only if $\bigcap_{i=1}^n V(\operatorname{in} f_i) = \emptyset$.

P r o o f. Let $d_i = \operatorname{ord} f_i$, $\Phi_i = \operatorname{in} f_i$ for $i = 1, \dots, n$. The inequality $\rho_o(F) \geq \Sigma_{i=1}^n(d_i - 1) + 1$ follows immediately from (2.1) since $\operatorname{ord}(\operatorname{Jac} F) \geq \Sigma_{i=1}^n(d_i - 1)$. We shall prove that the condition $\bigcap_{i=1}^n V(\Phi_i) = \emptyset$ implies the estimate $\rho_o(F) \leq \Sigma_{i=1}^n(d_i - 1) + 1$. According to (1.1) we have $\mu_o(F) = \prod_{i=1}^n d_i$ and, by (1.2), we can write $z_i^\mu = a_{i1}f_1 + \dots + a_{in}f_n$ with $\operatorname{ord}(a_{ij}f_j) \geq d_1 \dots d_n = \mu$, so $\operatorname{ord}(\det(a_{ij})) \geq \Sigma_{j=1}^n(\mu - d_j)$. By applying Lemma 2.2 to the germ $G = (z_1^\mu, \dots, z_n^\mu)$, for which $\rho_o(G) = n(\mu - 1) + 1$, we get

$$\rho_o(F) \leq n(\mu - 1) + 1 - \sum_{j=1}^{n}(\mu - d_j) = \sum_{j=1}^{n}((d_j - 1) + 1.$$

Now, let us consider a germ $F = (f_1, \dots, f_n)$ such that $\rho_o(F) = \Sigma_{j=1}^n(d_j - 1) + 1$. We have to show that $\bigcap_{i=1}^n V(\Phi_i) = \emptyset$. If we suppose, to the contrary, that $\bigcap_{i=1}^n V(\Phi_i) \neq \emptyset$ then, by a classical property of homogeneous polynomials (cf. [8] for a simple, direct proof of this property), the Jacobian $\mathrm{Jac}\,\Phi$ belongs to the ideal generated by $\Phi_1,\dots,\Phi_n$ and we can write $\mathrm{Jac}\Phi = A_1\Phi_1 + \dots + A_n\Phi_n$, where A_i is a homogeneous form of degree $\Sigma_{j=1}^n (d_j-1) - d_i$ or $A_i = 0$. Therefore

$$\mathrm{ord}(\mathrm{Jac}\Phi - A_1f_1 - \dots - A_nf_n) = \mathrm{ord}(A_1(\Phi_1 - f_1) + \dots + A_n(\Phi_n - f_n))$$
$$\geq \sum_{j=1}^{n} (d_j - 1) + 1 = \rho_o(F)$$

and we get $\mathrm{Jac}\Phi \in I(F)$. On the other hand, a simple calculation shows that

$$\mathrm{Jac}F = \mathrm{Jac}\Phi + \text{monomials of order} \geq \sum_{j=1}^{n} (d_j - 1) + 1.$$

Consequently, we get $\mathrm{Jac}F \in I(F)$, which is a contradiction with (1.3).

Let us present now a simple proof of Bertini's theorem in the form given by Tsikh (cf. [1], [9]).

THEOREM 2.4 (Bertini's theorem). *Let* $F = (f_1,\dots,f_n)$ *be a finite germ. Suppose that the set* $\bigcap_{i\neq i_o} V(\mathrm{in}f_i)$ *is finite for some* $i_o \in \{1,\dots,n\}$. *Then*

$$\rho_o(F) \leq \mu_o(F) - \prod_{j=1}^{n} \mathrm{ord}f_j + \sum_{j=1}^{n} (\mathrm{ord}f_j - 1) + 1.$$

Proof. We can assume that $i_o = n$. There is a linear form L such that $\bigcap_{i=1}^{n-1} V\cdot\mathrm{in}f_i) \cap V(L) = \emptyset$. Let $G = (f_1,\dots,f_{n-1},L^\mu)$, where $\mu = \mu_o(F)$. By Theorem 2.3 we have

$$\rho_o(G) = \sum_{i=1}^{n-1} (\mathrm{ord}f_i - 1) + \mu.$$

According to (1.2) we can write $L^\mu = a_1f_1 + \dots + a_nf_n$ with $\mathrm{ord}(a_jf_j) \geq \prod_{i=1}^n \mathrm{ord}f_i$. Hence, by (2.2), we get

$$\rho_o(F) \leq \rho_o(G) - \mathrm{ord}a_n \leq \sum_{i=1}^{n-1} (\mathrm{ord}f_i - 1) + \mu - \prod_{i=1}^{n} \mathrm{ord}f_i + \mathrm{ord}f_n$$

$$= \mu - \prod_{i=1}^{n} \mathrm{ord}f_i + \sum_{i=1}^{n} (\mathrm{ord}f_i - 1) + 1.$$

Bertini's estimates cannot be strengthened. Indeed, letting

$$F = (z_1^{d_1} + z_2^{d_1+\mu-d_1d_2}, z_1 z_2^{d_2-1}) (d_1, d_2 \geq 1, \ \mu \geq d_1 d_2)$$

we see that

$$\operatorname{ord} f_1 = d_1, \quad \operatorname{ord} f_2 = d_2, \quad \mu_o(F) = \mu, \quad \rho_o(F) = \mu - d_1 d_2 + d_1 + d_2 - 1$$

(one checks that $z_2^{\mu - d_1 d_2 + d_1 + d_2 - 2} \notin I(F)$). Similar examples can also be constructed for $n > 2$ (cf. example given in [7]). We complete this section with an application of (2.3) and (2.4).

<u>COROLLARY 2.5</u>. <u>Suppose that</u> F <u>satisfies the assumptions of</u> (2.4). <u>Let</u> $\mu_o(F) = \prod_{i=1}^n \operatorname{ord} f_i + 1$. <u>Then</u> $\rho_o(F) = \Sigma_{i=1}^n (\operatorname{ord} f_i - 1) + 2$.

<u>Proof</u>. By (1.1) we have $\bigcap_{i=1}^n V(\operatorname{in} f_i) \neq \emptyset$ so, by (2.3), we get $\rho_o(F) \geq \Sigma_{i=1}^n (\operatorname{ord} f_i - 1) + 2$. The equality follows from Bertini's theorem.

3. Concluding remarks

E. Bertini proved Theorem 2.4 in [1] (for $n = 2$) and in [2] (for $n \geq 2$), however his proof given in [2] contains some obscure points. I do not know how to prove (2.4) without additional assumptions concerning the initial forms. Note here that (2.4) holds true under the assumption that the set $\bigcap_{i=1}^n V(\operatorname{in} f_i)$ is finite. Indeed, we have the following

<u>PROPOSITION 3.1</u>. <u>Let</u> $F = (f_1, \ldots, f_n)$ <u>be a finite germ such that the set</u> $\bigcap_{i=1}^n V(\operatorname{in} f_i)$ <u>is finite</u>. <u>Then there exist power series</u> $g_1, \ldots, g_n$ <u>such that</u> (i) $g_1, \ldots, g_n$ <u>generate the ideal</u> $I(F)$, (ii) $\operatorname{ord} g_i = \operatorname{ord} f_i$ <u>for</u> $i = 1, \ldots, n$, <u>and</u> (iii) <u>there exists</u> $i_o \in \{1, \ldots, n\}$ <u>such that the set</u> $\bigcap_{i \neq i_o} V(\operatorname{in} g_i)$ <u>is finite</u>.

P r o o f. Let us keep the notation introduced in the proof of (2.3). Assume that $q_i = d_i - d_n \geq 0$ for $i = 1,\dots,n$ and put $V = \bigcap_{i=1}^{n-1} V(\Phi_i)$. From the assumption it follows that $\dim V \leq 1$, so there is a linear form L such that $V \cap V(L)$ is finite. Let $(a_1,\dots,a_{n-1})$ be the regular value (in the sense of Sard) of the mapping

$$(\Phi_1/L^{q_1}\Phi_n,\dots,\Phi_{n-1}/L^{q_{n-1}}\Phi_n) : \mathbb{C}\mathbb{P}^{n-1} \setminus V(L\Phi_n) \to \mathbb{C}^{n-1}.$$

We check easily that the set $\bigcap_{i=1}^{n-1} V(\Phi_i - a_i L^{q_i}\Phi_n)$ is finite, so it suffices to put

$$g_i = f_i - a_i L^{q_i} f_n \quad \text{for} \quad i = 1,\dots,n-1 \quad \text{and} \quad g_n = f_n.$$

Now, let $F = (f_1,\dots,f_n)$ be a finite germ such that the set $\bigcap_{i=1}^{n} V(\operatorname{in} f_i)$ is finite and let $G = (g_1,\dots,g_n)$, where g_i are the series like in (3.1). From $I(F) = I(G)$ it follows that $\mu_o(F) = \mu_o(G)$ and $\rho_o(F) = \rho_o(G)$. Hence we get Bertini's estimate for $\rho_o(F)$ by applying (2.4) to G.

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O WYKŁADNIKU NOETHERA

Streszczenie

Niech $F=(f_1,\dots,f_n)$ będzie kiełkiem odwzorowania holomorficznego o zerze izolowanym w punkcie $0\in\mathbb{C}^n$. Wykładnikiem Noethera $\rho_0(F)$ kiełka F nazywamy najmniejszą liczbę całkowitą $\rho>0$ taką, że każdy kiełek rzędu większego lub równego ρ leży w ideale generowanym przez $f_1,\dots,f_n$.

Celem tej pracy jest oszacowanie $\rho_0(F)$ w zależności od rzędów $\operatorname{ord} f_i$.