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A PROOF OF PALAMODOV'S THEOREM

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Abstract. We give a simple proof based on the Weierstrass Preparation Theorem of the following result due to V. P. Palamodov: if $f_1, \dots, f_n$ is a system of parameters of the formal power series ring $S = K[[x_1, \dots, x_n]]$ then S is a finitely generated free module over $R = K[[f_1, \dots, f_n]]$.

1. Introduction. Let $K[[\vec{x}]]$ be the ring of formal power series in n variables $\vec{x} = (x_1, \dots, x_n)$ with coefficients in a field K of arbitrary characteristic. For any sequence $\vec{f} = (f_1, \dots, f_n) \in K[[\vec{x}]]^n$ of power series without constant term we put $K[[\vec{f}]] = \{g \circ \vec{f} = g(f_1, \dots, f_n) : g \in K[[\vec{y}]], \vec{y} = (y_1, \dots, y_n)\}$. Then $K[[\vec{f}]]$ is a subring of $K[[\vec{x}]]$.

A sequence $\vec{f} = (f_1, \dots, f_n) \in K[[\vec{x}]]^n$ of power series without constant term is said to be a system of parameters (s.o.p.) if the ideal $I(\vec{f}) = (f_1, \dots, f_n)K[[\vec{x}]]$ generated by $(f_1, \dots, f_n)$ in $K[[\vec{x}]]$ is of finite codimension. We call $\mu = \dim_K K[[\vec{x}]]/I(\vec{f})$ the multiplicity of $\vec{f}$. Let us recall

Theorem 1.1 (The Generalized Weierstrass Preparation Theorem). *Let $\vec{f} = (f_1, \dots, f_n) \in K[[\vec{x}]]^n$ be a s.o.p. and let $e_0, e_1, \dots, e_{\mu-1}$ be a basis* mod *$I(\vec{f})$ i.e. a sequence of power series such that the images of $e_0, e_1, \dots, e_{\mu-1}$ under the natural epimorphism $K[[\vec{x}]] \to K[[\vec{x}]]/I(\vec{f})$ form a K-linear basis of $K[[\vec{x}]]/I(\vec{f})$. Then for every power series $g \in K[[\vec{x}]]$ there exist power series $g_0, g_1, \dots, g_{\mu-1} \in K[[\vec{y}]]$ such that*

$$g = \sum_{i=0}^{\mu-1} (g_i \circ \vec{f}) e_i.$$

The above version of the Weierstrass Preparation Theorem appeared first in Cartan's Seminar [**5**] in the following form: 'an algebra homomorphism of formal (analytic) algebras is finite if and only if it is quasi-finite.'

It was popularised by Malgrange in his famous monograph [**7**] (see also [**4**]). The formulation cited above is due to Arnold (see [**2**]). The case of formal series considered by us is easy to prove: we may write for any $g \in K[[\vec{x}]]$: $g = \sum_{i=0}^{\mu-1} c_i e_i + \sum_{j=1}^{n} g^{(j)} f_j$ with $c_i \in K$. Writing the same formula for $g^{(j)}$ and repeating the procedure we get the representation of g stated in Theorem 1.1 (see [**2**], Chapter 1 for more details). The Division Theorem and the Weierstrass Preparation Theorem are direct consequences of (1.1).

The Palamodov's Theorem [**10**] (§ 3, Theorem 2) tells us that the coefficients $g_i = g_i(\vec{y}), i = 0, 1, \dots, \mu - 1$ in the Generalized Preparation Theorem are uniquely determined by g:

Theorem 1.2 (Palamodov's Theorem). *Let $\vec{f} = (f_1, \dots, f_n) \in K[[\vec{x}]]^n$ be a s.o.p. and let $e_0, e_1, \dots, e_{\mu-1}$ be a basis* mod $I(\vec{f})$. *Then we have for any sequence of power series $g_0, g_1, \dots, g_{\mu-1} \in K[[\vec{y}]]$:*

$$\sum_{i=0}^{\mu-1} (g_i \circ \vec{f}) e_i = 0 \quad \Rightarrow \quad g_i = 0 \textit{ in } K[[\vec{y}]] \textit{ for } i = 0, 1, \dots, \mu - 1.$$

The aim of this note is to give a simple proof of Theorem 1.2 based on the Weierstrass Preparation Theorem. We will assume that **the field K is infinite**.

Our proof is given in Section 5. The original Palamodov's Theorem uses homological algebra and provides a Tor-criterion for finite modules over rings of convergent power series with coefficients in $\mathbb{C}$ (see [**10**] and Orlik's survey [**9**] for different proofs of Palamodov's result).

Note that Theorems 1.1 and 1.2 imply

Theorem 1.3. *Let $\vec{f}$ be a s.o.p. with multiplicity $\mu = \dim_K K[[\vec{x}]]/I(\vec{f})$. Then $K[[\vec{x}]]$ is a finitely generated free module over $K[[\vec{f}]]$ of rank μ.*

In [**8**] (Appendix B, Problem 3) Milnor indicates that Theorem 1.3 for $K = \mathbb{C}$ can be deduced from the coherence theorem for direct images under finite maps (a particular case of Grauert's Theorem [**3**]).

Let $(K[[\vec{x}]] : K[[\vec{f}]])$ denote the degree of the field of fractions of $K[[\vec{x}]]$ over the field of fractions of $K[[\vec{f}]]$. A direct corollary of (1.3) is

Theorem 1.4. *If $\vec{f}$ is a s.o.p. with multiplicity μ then $\mu = (K[[\vec{x}]] : K[[\vec{f}]])$.*

Theorem 1.4 implies that the algebraic multiplicity is equal to the covering (geometric) multiplicity (see [**9**], p. 419, Theorem 5.13 and [**6**], pp. 258–259). Note that a direct proof of this fact due to Kouchnirenko is outlined in [**2**].

2. Parameters in power series rings. In this section we prove some properties of parameters that we need in the proof of Palamodov's Theorem.

LEMMA 2.1. *Let $\vec{f} = (f_1, \ldots, f_n) \in K[[\vec{x}]]^n$ be a sequence of power series without constant term. Then the following three conditions are equivalent*

(i) *$\vec{f}$ is a s.o.p. in $K[[\vec{x}]]$,*
(ii) *$K[[\vec{x}]]$ is a finite $K[[\vec{f}]]$-module,*
(iii) *$K[[\vec{x}]]$ is integral over $K[[\vec{f}]]$.*

Proof. Implication (i) $\Rightarrow$ (ii) follows from the Generalized Weierstrass Preparation Theorem. Any finite extension of rings is integral, hence we get (ii) $\Rightarrow$ (iii). To prove that (iii) $\Rightarrow$ (i) consider the equations of integral dependence for x_i: $x_i^{m_i} + a_{i,1}(\vec{f})x_i^{m_i-1} + \cdots + a_{i,m_i}(\vec{f}) = 0$ $(i = 1, \ldots, n)$ of minimal degree $m_i > 0$. Using the Weierstrass Preparation Theorem we check that the power series $a_{i,j}$ are without constant term so that $a_{i,j}(\vec{y}) \in (\vec{y})K[[\vec{y}]]$. Therefore $a_{i,j}(\vec{f}) \in I(\vec{f})$ and $x_i^{m_i} = -a_{i,1}(\vec{f})x_i^{m_i-1} - \cdots - a_{i,m_i}(\vec{f}) \equiv 0 \bmod I(\vec{f})$ for $i = 1, \ldots, n$. Therefore $I(\vec{f})$ like the ideal $(x_1^{m_1}, \ldots, x_n^{m_n})\,K[[\vec{x}]]$ is of finite codimension. □

LEMMA 2.2. *If $\vec{f} = (f_1, \ldots, f_n) \in K[[\vec{x}]]^n$ and $\vec{g} = (g_1, \ldots, g_n) \in K[[\vec{y}]]^n$ are s.o.p. then $\vec{g} \circ \vec{f} = (g_1(f_1, \ldots, f_n), \ldots, g_n(f_1, \ldots, f_n)) \in K[[\vec{x}]]^n$ is a s.o.p.*

Proof. The extensions $K[[\vec{x}]] \supset K[[\vec{f}]]$ and $K[[\vec{f}]] \supset K[[\vec{g} \circ \vec{f}]]$ are finite by the Generalized Weierstrass Preparation Theorem. Therefore the extension $K[[\vec{x}]] \supset K[[\vec{g} \circ \vec{f}]]$ is finite and $\vec{g} \circ \vec{f}$ is a s.o.p. by Lemma 2.1. □

COROLLARY TO LEMMA 2.2. *If $\vec{f} = (f_1, \ldots, f_n) \in K[[\vec{x}]]^n$ is a s.o.p. then for any integers $m_1, \ldots, m_n > 0$ the sequence $(f_1^{m_1}, \ldots, f_n^{m_n}) \in K[[\vec{x}]]^n$ is also a s.o.p.*

3. Exchange Property. A power series $P \in K[[\vec{x}]]$ is x_n-regular of order $k > 0$ if $\operatorname{ord} P(0, \ldots, 0, x_n) = k$. Let $n > 1$. For any $c = (c_1, \ldots, c_{n-1}) \in K^{n-1}$ we put $\sigma_c(P) = P(x_1 + c_1x_n, \ldots, x_{n-1} + c_{n-1}x_n, x_n)$ for $P \in K[[\vec{x}]]$. Suppose that K is an infinite field. The following lemma is well-known.

LEMMA 3.1. *Let $P \in K[[\vec{x}]]$ be a non-zero power series in $n > 1$ variables without constant term. Then there is a non-zero polynomial $Q = Q(\vec{z})$, $\vec{z} = (z_1, \ldots, z_{n-1})$ such that the power series $\sigma_c(P)(0, \ldots, 0, x_n)$ is of order* $\operatorname{ord} P$ *if and only if $Q(\vec{c}) \neq 0$.*

PROOF. (see [**4**], Chapter I, Theorem 3). Let $k = \operatorname{ord} P$ and write $P = P_k + P_{k+1} + \cdots$ where P_i are homogeneous forms of degree i (or $P_i = 0$). Then $\sigma_c(P)(0, \ldots, 0, x_n) = P_k(c_1, \ldots, c_{n-1}, 1)x_n^k +$ *higher order terms.* We put $Q(\vec{z}) = P_k(z_1, \ldots, z_{n-1}, 1)$. □

PROPOSITION 3.2 (The Exchange Property). *Let $\vec{f} = (f_1, \dots, f_n) \in K[[\vec{x}]]^n$ $(n > 1)$ be a s.o.p. and let $g \in K[[\vec{x}]]$ be a nonzero power series without constant term. Then there is a non-zero polynomial $Q = Q(\vec{z})$, $\vec{z} = (z_1, \dots, z_{n-1})$ such that if $Q(c) \neq 0$ then the sequence $f_1 - c_1 f_n, \dots, f_{n-1} - c_{n-1} f_n, g$ is a s.o.p. in $K[[\vec{x}]]$.*

PROOF. By Lemma 2.1 the ring $K[[\vec{x}]]$ is integral over $K[[\vec{f}]]$. Let $g^m + P_1(\vec{f})g^{m-1} + \dots + P_m(\vec{f}) = 0$ be the equation of integral dependence for g of minimal degree $m > 0$. Then $P_m(\vec{y}) \neq 0$ in $K[[\vec{y}]]$ and $P_m(0) = 0$. By Lemma 3.1 there is a non-zero polynomial $Q = Q(\vec{z})$ such that $\sigma_c(P_m) = P_m(y_1 + c_1 y_n, \dots, y_{n-1} + c_{n-1} y_n, y_n)$ is y_n-regular of order $k = \operatorname{ord} P_m$ if $Q(c) \neq 0$. Let $P(\vec{y}, t) = t^m + P_1(\vec{y})t^{m-1} + \dots + P_m(\vec{y}) \in K[[\vec{y}]][t]$ and $\sigma_c(P)(\vec{y}, t) = t^m + \sigma_c(P_1)(\vec{y})t^{m-1} + \dots + \sigma_c(P_m)(\vec{y})$. Fix $c \in K^{n-1}$ such that $Q(c) \neq 0$. Since $\sigma_c(P)(0, y_n, 0) = \sigma_c(P_m)(0, y_n)$ is of order $k > 0$ then we get by the Weierstrass Preparation Theorem

$$\sigma_c(P) = \big(y_n^k + Q_1(y_1, \dots, y_{n-1}, t)y_n^{k-1} + \dots + Q_k(y_1, \dots, y_{n-1}, t)\big)U(\vec{y}, t) \tag{1}$$

in $K[[\vec{y}, t]]$ where $Q_i(0) = 0$ for $i = 1, \dots, k$ and $U(0, 0) \neq 0$. Let $f_i^{(c)} = f_i - c_i f_n$ for $i = 1, \dots, n-1$ and $f_n^{(c)} = f_n$. Then

$$\sigma_c(P)(f_1^{(c)}, \dots, f_n^{(c)}, g) = P(\vec{f}, g) = 0 \text{ in } K[[\vec{x}]] \tag{2}$$

and by (1) and (2) we get

$$f_n^k + Q_1(f_1^{(c)}, \dots, f_{n-1}^{(c)}, g)f_n^{k-1} + \dots + Q_k(f_1^{(c)}, \dots, f_{n-1}^{(c)}, g) = 0. \tag{3}$$

Since Q_i $(i = 1, \dots, k)$ are without constant term we get from (3):

$$f_n^k \equiv 0 \bmod (f_1^{(c)}, \dots, f_{n-1}^{(c)}, g)K[[\vec{x}]]. \tag{4}$$

Since $(f_1^{(c)}, \dots, f_{n-1}^{(c)}, f_n)K[[\vec{x}]] = (f_1, \dots, f_n)K[[\vec{x}]]$ is of finite codimension the sequence $f_1^{(c)}, \dots, f_{n-1}^{(c)}, f_n^k$ is a s.o.p. by Corollary to Lemma 2.2.

By (4) we have $(f_1^{(c)}, \dots, f_{n-1}^{(c)}, f_n^k)K[[\vec{x}]] \subset (f_1^{(c)}, \dots, f_{n-1}^{(c)}, g)K[[\vec{x}]]$ and $f_1^{(c)}, \dots, f_{n-1}^{(c)}, g$ is a s.o.p. □

REMARK 3.3. Another version of the exchange property (for graded polynomial rings, based on dimension theory) is given in [**12**], Chapter 2, Section on the Cohen–Maclaulay property. For the underlying geometric idea see Shafarevich treatment of intersection theory [**11**], Chapter 4.

4. Preparatory lemmas. A sequence of power series $\vec{f} = (f_1, \dots, f_n) \in K[\vec{x}]]^n$ without constant term will be called Palamodov's sequence (in short: P-sequence) if there exist power series $e_0, e_1, \dots, e_{m-1} \in K[[\vec{x}]]$ $(m > 0)$ such that

(P_1) for any $g \in K[[\vec{x}]]$ there is a sequence $g_0, g_1, \dots, g_{m-1} \in K[[\vec{y}]]$ such that $g = \sum_{i=0}^{m-1} (g_i \circ \vec{f}) e_i$ in $K[[\vec{x}]]$,

(P_2) if $g_0, g_1, \dots, g_{m-1} \in K[[\vec{y}]]$ are such that $\sum_{i=0}^{m-1} (g_i \circ \vec{f}) e_i = 0$ in $K[[\vec{x}]]$ then $g_i = 0$ for $i = 1, \dots, m-1$ in $K[[\vec{y}]]$.

Lemma 4.1. *Suppose that $\vec{f} \in K[[\vec{x}]]^n$ is a P-sequence. Then*

(a) *the series $f_1, \dots, f_n \in K[[\vec{x}]]$ are analytically independent i.e. for any power series $g_0 \in K[[\vec{y}]]$:*

$$g_0(f_1, \dots, f_n) = 0 \text{ in } K[[\vec{x}]] \ \Rightarrow \ g_0 = 0 \text{ in } K[[\vec{y}]].$$

(b) *For any $k > 0$: f_k is not a zero-divisor* $\bmod (f_1, \dots, f_{k-1}) K[[\vec{x}]]$ *i.e. for any $g \in K[[\vec{x}]]$ if $g f_k \equiv 0 \mod (f_1, \dots, f_{k-1}) K[[\vec{x}]]$ then $g \equiv 0 \mod (f_1, \dots, f_{k-1}) K[[\vec{x}]]$.*

Proof. The first part of the lemma is a direct consequence of (P_2). To prove the second part we write by (P_1) $g = \sum_{i=0}^{m-1} (g_i \circ \vec{f}) e_i$. From (P_2) it follows that the above representation is unique.

Let $k > 0$.

Claim. $g \in (f_1, \dots, f_{k-1}) K[[\vec{x}]]$ if and only if $g_i \circ \vec{f} \in (f_1, \dots, f_{k-1}) K[[\vec{f}]]$ for $i = 0, 1, \dots, m-1$.

Proof of the claim. If $g_i \circ \vec{f} \in (f_1, \dots, f_{k-1}) K[[\vec{f}]]$ then obviously $g = \sum_{i=0}^{m-1} (g_i \circ \vec{f}) e_i \in (f_1, \dots, f_{k-1}) K[[\vec{x}]]$. □

Suppose that $g \in (f_1, \dots, f_{k-1}) K[[\vec{x}]]$. Then we can write $g = \sum_{j=1}^{k-1} h_j f_j = \sum_{j=1}^{k-1} \left(\sum_{i=0}^{m-1} (h_{j,i} \circ \vec{f}) \vec{e_i} \right) f_j = \sum_{i=0}^{m-1} \left(\sum_{j=1}^{k-1} (h_{j,i} \circ \vec{f}) f_j \right) e_i$.

On the other hand $g = \sum_{i=0}^{m-1} (g_i \circ \vec{f}) e_i$ and by the uniqueness of representation (P_1) we get $g_i \circ \vec{f} = \sum_{j=1}^{k-1} (h_{j,i} \circ \vec{f}) f_j$ for $i = 0, 1, \dots, m-1$.

To check Part (b) of the lemma suppose that $g f_k \equiv 0 \bmod (f_1, \dots, f_{k-1}) K[[\vec{x}]]$. Then $g f_k = \sum_{i=0}^{m-1} ((g_i \circ \vec{f}) f_k) e_i$ and by the claim $f_k (g_i \circ \vec{f}) \in (f_1, \dots, f_{k-1}) K[[\vec{f}]]$. Hence $y_k g_i(\vec{y}) \in (y_1, \dots, y_{k-1}) K[[\vec{y}]]$ by (a) and $g_i(\vec{y}) \in (y_1, \dots, y_{k-1}) K[[\vec{y}]]$ since y_k does not divide zero $\bmod (y_1, \dots, y_{k-1}) K[[\vec{y}]]$. This implies $g_i(\vec{f}) \in (f_1, \dots, f_{k-1}) K[[\vec{f}]]$ and $g = \sum_{i=0}^{m-1} (g_i \circ \vec{f}) e_i \in (f_1, \dots, f_{k-1}) K[[\vec{x}]]$. □

If $n > 1$ we put $\vec{x}' = (x_1, \dots, x_{n-1})$ and for any $f \in K[[\vec{x}]]$ we let $f' = f(x_1, \dots, x_{n-1}, 0) \in K[[\vec{x}']]$.

LEMMA 4.2. *Let* $f_1, \dots, f_{n-1} \in K[[\vec{x}]]$ $(n > 1)$ *be power series such that* $f'_1, \dots, f'_{n-1}$ *is a P-sequence. Then* $f_1, \dots, f_{n-1}$ *and* $f_n = x_n$ *form a P-sequence.*

PROOF. Let $\vec{f'} = (f'_1, \dots, f'_{n-1})$ and take a sequence $(e_i)_{i=0,\dots,m-1}$, $e_i \in K[[\vec{x}']]$ such that

(P'_1) for any $h \in K[[\vec{x}']]$ there is a sequence $h_0, \dots, h_{m-1} \in K[[\vec{y}']]$, $\vec{y}' = (y_1, \dots, y_{n-1})$ such that $h = \sum_{i=0}^{m-1}(h_i \circ \vec{f'})e_i$,

(P'_2) if $h_0, \dots, h_{m-1} \in K[[\vec{y}']]$ are such that $\sum_{i=0}^{m-1}(h_i \circ \vec{f'})e_i = 0$ then $h_i = 0$ for $i = 1, \dots, m-1$ in $K[[\vec{y}]]$.

Let $\vec{f} = (f_1, \dots, f_{n-1}, x_n)$. We will check properties (P_1) and (P_2). Fix $g \in K[[\vec{x}]]$ and write $g = \sum_{k=0}^{\infty} h_k x_n^k$ where $h_k \in K[[\vec{x}']]$ for $k = 0, 1, \dots$.

By (P'_1) we get $h_k = \sum_{i=0}^{m-1}(h_{k,i} \circ \vec{f'})e_i$.
Therefore $g = \sum_{i=0}^{m-1}\left(\sum_{k=0}^{\infty}(h_{k,i} \circ \vec{f'})x_n^k\right)e_i$. Let $g_i(\vec{y}) = \sum_{k=0}^{\infty} h_{k,i}(\vec{y}')y_n^k$ for $i = 0, 1, \dots, m-1$. Then $g = \sum_{i=0}^{m-1}(g_i \circ \vec{f})e_i$ which proves (P_1).

Suppose that $g_0, \dots, g_{m-1} \in K[[\vec{y}]]$ are such that $\sum_{i=0}^{m-1}(g_i \circ \vec{f})e_i = 0$ i. e. $\sum_{i=0}^{m-1} g_i(f_1(\vec{x}), \dots, f_{n-1}(\vec{x}), x_n)e_i = 0$ in $K[[\vec{x}]]$.
Then $\sum_{i=0}^{m-1} g_i(f_1(\vec{x}', 0), \dots, f_{n-1}(\vec{x}', 0), 0)e_i = 0$ and $g_i(y_1, \dots, y_{m-1}, 0) = 0$ for $i = 0, \dots, m-1$ in $K[[\vec{y}']]$ by (P'_2). Therefore we may write $g_i = y_n\tilde{g}_i$ for $i = 0, \dots, m-1$. Repeating this reasoning we check that $g_i \equiv 0 \pmod{y_n^q}$ for all $q \geq 0$ and $g_i = 0$ in $K[[\vec{y}]]$ for $i = 0, \dots, m-1$. This proves (P_2). □

5. Proof of Palamodov's Theorem. We omit the easy proof of Palamodov's Theorem in the case $n = 1$. Suppose that $n > 1$ and Palamodov's Theorem is true for s.o.p. in the ring of formal power series in $n-1$ variables.

Fix a s.o.p. $\vec{f} = (f_1, \dots, f_n) \in K[[\vec{x}]]^n$ and suppose that there exist two finite families of power series $(g_i)_{i \in I}$, $g_i \in K[[\vec{y}]]$ and $(e_i)_{i \in I}$, $e_i \in K[[\vec{x}]]$, $I \neq \emptyset$ such that

$$g_i \neq 0 \text{ for } i \in I \text{ and } \sum_{i \in I}(g_i \circ \vec{f}_i)e_i = 0 \text{ in } K[[\vec{x}]]. \tag{5}$$

We will check that the family $(e_i)_{i \in I}$ is K-linearly dependent $\operatorname{mod} I(\vec{f})$. This will prove Palamodov's Theorem for the case of n variables.

From (5) we get

$$\sum_{i \in I} g_i(0)e_i \equiv 0 \mod I(\vec{f}). \tag{6}$$

If there is $i \in I$ such that $g_i(0) \neq 0$ then the family $(e_i)_{i \in I}$ is K-linearly dependent $\operatorname{mod} I(\vec{f})$ and the assertion is proved. Therefore we assume in the sequel that

$$g_i(0) = 0 \quad \text{for all } i \in I. \tag{7}$$

Let $c = (c_1, \dots, c_{n-1}) \in K^{n-1}$. In the notation introduced in Section 3 we get

$$\sum_{i\in I} \big(\sigma_c(g_i) \circ \vec{f}^{(c)}\big) e_i = 0 \quad \text{in } K[[\vec{x}]] \tag{8}$$

and (we recall that $f_n^{(c)} = f_n$)

$$\sum_{i\in I} \big(\sigma_c(g_i)(0, f_n)\big) e_i \equiv 0 \quad \mod (f_1^{(c)}, \dots, f_{n-1}^{(c)}) K[[\vec{x}]]. \tag{9}$$

By Lemma 3.1 and Proposition 3.2 we can choose $c \in K^{n-1}$ such that

$$\sigma_c(g_i)(0, y_n) \text{ is of order } \operatorname{ord} g_i \text{ for all } i \in I, \tag{10}$$

$$f_1^{(c)}, \dots, f_{n-1}^{(c)}, x_n \text{ is a s.o.p. in } K[[\vec{x}]]. \tag{11}$$

We claim that

$$f_n \quad \text{is not a zero-divisor} \quad \mod (f_1^{(c)}, \dots, f_{n-1}^{(c)}) K[[\vec{x}]]. \tag{12}$$

To check (12) let $J = (f_1^{(c)}, \dots, f_{n-1}^{(c)}) K[[\vec{x}]]$ and suppose that $g f_n \equiv 0 \bmod J$ for a series $g \in K[[\vec{x}]]$. Since $f_1^{(c)}, \dots, f_{n-1}^{(c)}, f_n$ is a s.o.p. there is a series $h \in K[[\vec{x}]]$ such that $x_n^q \equiv h f_n \bmod J$ for an integer $q > 0$. Now we get $g h f_n \equiv 0 \bmod J$ and $g x_n^q \equiv 0 \bmod J$.

Since Palamodov's Theorem holds in $K[[\vec{x}']]$ the sequence $f_1^{(c)}(x', 0), \dots, f_{n-1}^{(c)}(x', 0)$ is a P-sequence and by Lemmas 4.2 and 4.1 $f_1^{(c)}, \dots, f_{n-1}^{(c)}, x_n$ is a P-sequence. Consequently x_n (and therefore x_n^q) is not a zero-divisor $\pmod J$. Thus we get $g \equiv 0 \bmod J$ and (12) is proved.

Let us put $r = \min\{ \operatorname{ord} g_i : i \in I \}$ and write $\sigma_c(g_i)(0, y_n) = y_n^r h_i(y_n)$ for $i \in I$. From (5) we get

$$f_n^r \sum_{i\in I} h_i(f_n) e_i \equiv 0 \quad \mod (f_1^{(c)}, \dots, f_{n-1}^{(c)}) K[[\vec{x}]]. \tag{13}$$

Hence by (12) we obtain

$$\sum_{i\in I} h_i(f_n) e_i \equiv 0 \bmod (f_1^{(c)}, \dots, f_{n-1}^{(c)}) K[[\vec{x}]]$$

and

$$\sum_{i\in I} h_i(0) e_i \equiv 0 \quad \mod I(\vec{f})$$

for $(f_1^{(c)}, \dots, f_{n-1}^{(c)}, f_n) K[[\vec{x}]] = I(\vec{f})$. By definition of r there is an $i \in I$ such that $h_i(0) \neq 0$ and we are done. □

Remark 5.1. We could extend our proof to the case of finite field K by replacing the linear change of coordinates $(x_1, \dots, x_n) \to (x_1 + c_1 x_n, \dots, x_{n-1} + c_{n-1} x_n, x_n)$ by the polynomial automorphism $(x_1, \dots, x_n) \to (x_1 + x_n^{p_1}, \dots, x_{n-1} + x_n^{p_{n-1}}, x_n)$.

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