Algebraic Dependence of Polynomials After O. Perron and Some Applications[1]

Arkadiusz Płoski [2]

Department of Mathematics, Technical University

Abstract. We give some applications of a classical theorem (Oskar Perron 1927) on algebraic dependence to Bézout's type estimations.

Keywords. Algebraic dependence, polynomial automorphism, Bézout's theorem

"The method of high-school algebra is powerful, beautiful, and accessible. So let us not be overwhelmed by the groups-rings-fields or the functional arrows of the other two algebras and thereby lose sight of the power of the explicit algorithmic processes..."

Shreeram S. Abhyankar

The aim of this note is to give an example that illustrates Shreeram S. Abhyankar's thesis quoted above. Other examples the reader will find in [1]. We will show how classical Perron's Theorem on algebraic dependence of polynomials may be used to get nontrivial results such as a bound on the degree of the inverse of a polynomial automorphism and a weak version of Bézout's Theorem. Oskar Perron published the theorem on algebraic dependence of polynomials in Algebra I (Die Grundlagen), Berlin 1927 and used it as a basis for his original treatment of elimination theory. Later on he gave a version of the theorem in the article "Beweis und Verschärfung eines Satzes von Kronecker" Math. Annalen 118 (1942), S. 441–448.

The fact that $n+1$ polynomials of n variables are algebraically dependent belongs to the very beginning of classical algebra and could have been known in the 18th century. Perron's Theorem gives a bound on the weight of the algebraic relation between polynomials that makes the classical theorem on algebraic dependence effective. To prove his result Perron uses an elimination procedure based on linear algebra and Kronecker's method of introducing and specializing new variables. We give a detailed presentation of Perron's proof in Section 2. Then we give some applications of Perron's Theorem to Bézout's type estimates. Further applications the reader will find in [3], [6], [8], [9]. Recently Z. Jelonek has given a proof of the effective Nullstellensatz based on Perron's Theorem (Z. Jelonek, *On the Łojasiewicz exponent and effective Nullstellensatz*, to appear).

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[2] Arkadiusz Płoski, AL. 1000 L PP 7, 25-314 Kielce, Poland. E-mail: matap@tu.kielce.pl.

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1. Perron's Theorem

Let $\mathbf{K}$ be a field. We will consider polynomials with coefficients in $\mathbf{K}$. The following theorem is due to O.Perron ([7], Satz 57, S.129).

Theorem 1.1 *Let $F_1, \ldots, F_{n+1} \in \mathbf{K}[X]$ be a sequence of $n+1$ nonconstant polynomials in n variables $X = (X_1, \ldots, X_n)$ and let $\deg F_i = d_i$ for $i = 1, \ldots, n+1$. Then there exists a nonzero polynomial $P = P(Y) \in \mathbf{K}[Y]$ in $n+1$ variables $Y = (Y_1, \ldots, Y_{n+1})$ such that*

(a) $P(F_1, \ldots, F_{n+1}) = 0$,
(b) *if* $\operatorname{weight} Y_i = d_i$ *for* $i = 1, \ldots, n+1$ *then* $\operatorname{weight} P \leq d_1 \ldots d_{n+1}$.

A polynomial P satisfying the assumptions (a) and (b) will be called Perron's relation between $F_1, \ldots, F_{n+1}$. Let us recall here that if $\operatorname{weight} Y_i = d_i$ for $i = 1, \ldots, n+1$ then $\operatorname{weight}(cY_1^{a_1} \ldots Y_{n+1}^{a_{n+1}}) = a_1 d_1 + \ldots + a_{n+1} d_{n+1}$ (if $c \in \mathbf{K}^*$) and the weight of a nonzero polynomial P is, by definition, equal to the maximum of weights of monomials appearing in P with a nonzero coefficient. Let $\Delta = \{(a_1, \ldots, a_{n+1}) \in \mathbf{N}^{n+1} : d_1 a_1 + \ldots + d_{n+1} a_{n+1} \leq d_1 \ldots d_{n+1}\}$. Then Perron's relation P between $F_1, \ldots, F_{n+1}$ can be written in the form

$$P(Y_1, \ldots, Y_{n+1}) = \sum_{(a_1, \ldots, a_{n+1}) \in \Delta} c_{a_1, \ldots, a_{n+1}} Y_1^{a_1} \ldots Y_{n+1}^{a_{n+1}} .$$

Note that

$$\deg P \leq \frac{\prod_{i=1}^{n+1} d_i}{\min_{i=1}^{n+1} d_i} \leq \left(\max_{i=1}^{n+1} d_i\right)^n .$$

The proof of Perron's theorem we give in Section 2 of this note. The problem of finding Perron's relation reduces to solving a system of linear homogeneous equations. To see this let us consider a collection of new variables

$$C = (C_{a_1, \ldots, a_{n+1}} : (a_1, \ldots, a_{n+1}) \in \Delta) .$$

Let $\Delta^* = \{(b_1, \ldots, b_n) \in \mathbf{N}^n : b_1 + \ldots + b_n \leq d_1 \ldots d_{n+1}\}$. Then

$$\sum_{(a_1, \ldots, a_{n+1}) \in \Delta} C_{a_1, \ldots, a_{n+1}} F_1^{a_1} \ldots F_{n+1}^{a_{n+1}} = \sum_{(b_1, \ldots, b_n) \in \Delta^*} L_{b_1, \ldots, b_n}(C) X_1^{b_1} \ldots X_n^{b_n}$$

in the ring $\mathbf{K}[C, X]$. Clearly the polynomials $L_{b_1, \ldots, b_n}(C)$ are linear and homogeneous and we have

Property 1.2 *A polynomial* $P = \sum_{(a_1,\dots,a_{n+1})\in\Delta} c_{a_1,\dots,a_{n+1}} Y_1^{a_1}\dots Y_{n+1}^{a_{n+1}}$ *is Perron's relation between* $F_1,\dots,F_{n+1}$ *if and only if the collection* $c = (c_{a_1,\dots,a_{n+1}} : (a_1,\dots,a_{n+1}) \in \Delta)$ *is a nonzero solution of the system of linear homogeneous equations* $L_{b_1,\dots,b_n}(C) = 0$, $(b_1,\dots,b_n) \in \Delta^*$.

In general Perron's relation is not uniquely determined by the given sequence of polynomials. If the sequence $F_1,\dots,F_{n+1}$ contains n algebraically independent polynomials then there is a unique irreducible polynomial (up to a constant factor in $\mathbf{K}^*$) $P_0 = P_0(Y_1,\dots,Y_{n+1})$ such that $P_0(F_1,\dots,F_{n+1}) = 0$. We call P_0 the minimal polynomial of $F_1,\dots,F_{n+1}$. Let us note the following simple

Corollary 1.3 *Suppose that the sequence* $F_1,\dots,F_{n+1}$ *contains* n *algebraically independent polynomials. Then the minimal polynomial of* $F_1,\dots,F_{n+1}$ *is Perron's relation between* $F_1,\dots,F_{n+1}$.

Proof. Let P be a Perron's relation between $F_1,\dots,F_{n+1}$. Then P_0 divides P and we get $\operatorname{weight} P_0 \le \operatorname{weight} P \le d_1\dots d_{n+1}$.

Corollary 1.4 *Suppose that the polynomials* $F_1,\dots,F_n \in \mathbf{K}[X]$ *of degree* $d_1,\dots,d_n > 0$ *are algebraically independent. Let* $G \in \mathbf{K}[Y_1,\dots,Y_n]$ *and* $H = G(F_1,\dots,F_n)$. *Then* $\operatorname{weight} G \le (\prod_{i=1}^n d_i)\deg H$.

Proof. Set $P(Y_1,\dots,Y_{n+1}) = Y_{n+1} - G(Y_1,\dots,Y_n)$. Then P is the minimal polynomial of $F_1,\dots,F_n,H$. By Corollary 1.3 we get $\operatorname{weight} P \le (\prod_{i=1}^n d_i)(\deg H)$. On the other hand $\operatorname{weight} P = \max\{\operatorname{weight} Y_{n+1}, \operatorname{weight} G\}$ and consequently we get $\operatorname{weight} G \le \operatorname{weight} P \le (\prod_{i=1}^n d_i)(\deg H)$.

2. Proof of Perron's Theorem

Let $F_1,\dots,F_n$ be polynomials in n variables $X = (X_1,\dots,X_n)$ of degree $d_1,\dots,d_n > 0$ with coefficients in an extension $\mathbf{L}$ of the field $\mathbf{K}$.

Proposition 2.1 *Suppose that the coefficients of these polynomials are algebraically independent over* $\mathbf{K}$. *Then for every nonconstant polynomial* $G = G(X) \in \mathbf{L}[X]$ *there exists a family of polynomials*

$$G_{r_1,\dots,r_n}(Y_1,\dots,Y_n), \quad 0 \le r_i < d_i \text{ for } i = 1,\dots,n$$

such that

(a) $G = \sum G_{r_1,\dots,r_n}(F_1,\dots,F_n) X_1^{r_1}\dots X_n^{r_n}$,
(b) *if* $\operatorname{weight} Y_i = d_i$ *for* $i = 1,\dots,n$ *then* $\operatorname{weight} G_{r_1,\dots,r_n} + r_1 + \dots + r_n \le \deg G$.

Proof. It suffices to check that for every $N > 0$ the family

$$F_1^{a_1}\dots F_n^{a_n} X_1^{r_1}\dots X_n^{r_n}$$

where $0 \le r_i < d_i$ for $i = 1,\dots,n$ and $\sum_{i=1}^n a_i d_i + \sum_{i=1}^n r_i \le N$ is a linear basis of the space $\mathbf{L}[X]_N = \{G \in \mathbf{L}[X] : \deg G \le N\}$. For any sequence $(l_1,\dots,l_n) \in \mathbf{N}^n$ we put

$$Q_{l_1,\dots,l_n} = F_1^{a_1}\dots F_n^{a_n} X_1^{r_1}\dots X_n^{r_n}$$

where a_i and r_i are determined by the conditions $l_i = a_i d_i + r_i$, $0 \le r_i < d_i$. Clearly the mapping $(l_1,\dots,l_n) \to (a_1,\dots,a_n,r_1,\dots,r_n)$ induces a bijection of the sets $\{(l_1,\dots,l_n) : \sum_{i=1}^n l_i \le N\}$ and $\{(a_1,\dots,a_n,r_1,\dots,r_n) : \sum_{i=1}^n a_i d_i + \sum_{i=1}^n r_i \le N$ and $0 \le r_i < d_i$ for $i = 1,\dots,n\}$. Therefore it suffices to show that the family $(Q_{l_1,\dots,l_n} : \sum_{i=1}^n l_i \le N)$ is a linear basis of $\mathbf{L}[X]_N$. This follows from the following observations:

1. $Q_{l_1,\dots,l_n} \in \mathbf{L}[X]$ is of degree $\sum_{i=1}^n l_i$,
2. the coefficients of $Q_{l_1,\dots,l_n}$ lie in the ring R of coefficients of polynomials $F_1,\dots,F_n$. If $G \to \bar{G}$ is a specialization of $R[X]$ in $\mathbf{K}[X]$ such that $\bar{F}_1 = X_1^{d_1},\dots,\bar{F}_n = X_n^{d_n}$ then $\bar{Q}_{l_1,\dots,l_n} = X_1^{l_1}\dots X_n^{l_n}$.
3. If D is the determinant of the family $(Q_{l_1,\dots,l_n} : \sum_{i=1}^n l_i \le N)$ with respect to the linear basis $(X_1^{k_1}\dots X_n^{k_n} : \sum_{i=1}^n k_i \le N)$ of $\mathbf{L}[X]_N$ (we consider $\mathbf{N}^n$ with lexicographic order) then $\bar{D} \ne 0$ and consequently $D \ne 0$.

Proposition 2.2 *Suppose that the coefficients of the polynomials $F_1,\dots,F_n$ are algebraically independent over* $\mathbf{K}$. *Let $d = \prod_{i=1}^n d_i$. Then for every polynomial F_{n+1} of degree $d_{n+1} > 0$ there exists a polynomial $P = Y_{n+1}^d + \sum_{i=1}^d P_i(Y_1,\dots,Y_n)Y_{n+1}^{d-i}$ such that $P(F_1,\dots,F_n,F_{n+1}) = 0$ and* weight $P \le \prod_{i=1}^{n+1} d_i$ *provided* weight $Y_i = d_i$ *for* $i = 1,\dots,n+1$.

Proof. Let $M_0 = 1$, ..., $M_{d-1} = X_1^{d_1-1}\dots X_n^{d_n-1}$ be a sequence of monomials $X_1^{r_1}\dots X_n^{r_n}$, $0 \le r_i < d_i$ for $i = 1,\dots,n$. By Proposition 2.1 there exist polynomials $P_{ij} = P_{ij}(Y_1,\dots,Y_n)$ such that

(a) $M_i F_{n+1} = \sum_{j=0}^{d-1} P_{ij}(F_1,\dots,F_n)M_j$ for $i = 0,\dots,d-1$,
(b) if weight $Y_i = d_i$ for $i = 1,\dots,n$ then weight $P_{ij} + \deg M_j \le \deg M_i + d_{n+1}$.

By Cramer's rule we get

(c) $\det(P_{ij}(F_1,\dots,F_n) - \delta_{ij}F_{n+1}) = 0$.

Let $P(Y_1,\dots,Y_{n+1}) = (-1)^d \det(P_{ij}(Y_1,\dots,Y_n) - \delta_{ij}Y_{n+1})$. Then $P = Y_{n+1}^d + \sum_{i=1}^d P_i(Y_1,\dots,Y_n)Y_{n+1}^{d-i}$ and $P(F_1,\dots,F_n,F_{n+1}) = 0$. Let weight $Y_{n+1} = d_{n+1}$. To estimate weight P set $\tilde{P}_{ij} = P_{ij} - \delta_{ij}Y_{n+1}$. By (b) we have weight $\tilde{P}_{ij} \le d_{n+1} + \deg M_i - \deg M_j$ and weight$(\pm\tilde{P}_{0,j_0}\dots\tilde{P}_{d-1,j_{d-1}}) \le (d_{n+1} + \deg M_0 - \deg M_{j_0}) + \dots + (d_{n+1} + \deg M_{d-1} - \deg M_{j_{d-1}}) = d_{n+1}d = \prod_{i=1}^{n+1} d_i$ for any permutation $(j_0,\dots,j_{d-1})$ of $(0,1,\dots,d-1)$. Consequently weight $P =$ weight$(\sum \pm P_{0,j_0}\dots P_{d-1,j_{d-1}}) \le \prod_{i=1}^{n+1} d_i$. From Proposition 2.2 it follows that Perron's Theorem is true for polynomials with algebraically independent coefficients. This implies the theorem in the general case. To see this let us fix a sequence of positive integers $d_1,\dots,d_{n+1} > 0$ and let $\Delta = \{(a_1,\dots,a_{n+1}) \in \mathbf{N}^{n+1} : \sum_{i=1}^{n+1} a_i d_i \le d_1\dots d_{n+1}\}$. For any sequence of polynomials $F_1,\dots,F_{n+1}$ with degree $d_1,\dots,d_{n+1}$ we define $M(F_1,\dots,F_{n+1})$ to be the matrix of the family $(F_1^{a_1}\dots F_{n+1}^{a_{n+1}} : (a_1,\dots,a_{n+1}) \in \Delta)$ with respect to the linear basis $(X_1^{k_1}\dots X_n^{k_n} : \sum_{i=1}^n k_i \le d_1\dots d_{n+1})$. Perron's Theorem is true for $F_1,\dots,F_{n+1}$ if and only if the polynomials $F_1^{a_1}\dots F_{n+1}^{a_{n+1}}$ of degree less than or equal to $\prod_{i=1}^{n+1} d_i$ are linearly dependent which is equivalent to the condition $\operatorname{rank} M(F_1, \ldots, F_{n+1}) < \operatorname{card} \Delta$.

If $F_1 \ldots, F_{n+1}$ are polynomials with algebraically independent coefficients then $\operatorname{rank} M(F_1, \ldots, F_{n+1}) < \operatorname{card} \Delta$ by Proposition 2.2. Take specializations $\bar{F}_1, \ldots, \bar{F}_{n+1}$ of the polynomials $F_1, \ldots, F_{n+1}$. Then $M(\bar{F}_1, \ldots, \bar{F}_{n+1}) = \overline{M(F_1, \ldots, F_{n+1})}$ and consequently $\operatorname{rank} M(\bar{F}_1, \ldots, \bar{F}_{n+1}) < \operatorname{card} \Delta$ which proves Perron's Theorem for $\bar{F}_1, \ldots, \bar{F}_{n+1}$.

3. Polynomial Automorphisms

A polynomial map over $\mathbf{K}$ is by definition a sequence of polynomials $F = (F_1, \ldots, F_n)$ with coefficients in $\mathbf{K}$. We say that F is a polynomial automorphism if there exist $G_1, \ldots, G_n \in \mathbf{K}[Y_1, \ldots, Y_n]$ such that $G_i(F_1(X), \ldots, F_n(X)) = X_i$ for $i = 1, \ldots, n$. We write $F^{-1} = (G_1, \ldots, G_n)$. For any polynomial mapping $F = (F_1, \ldots, F_n)$ we put $\deg F = \max_{i=1}^{n} (\deg F_i)$.

As a first application of Perron's Theorem we give a proof of the following known result (see [2], [4], [9], [10]).

Theorem 3.1 (on the degree of the inverse of an automorphism)
Let $F = (F_1, \ldots, F_n)$ be a polynomial automorphism over $\mathbf{K}$. Then

$$\deg F^{-1} \le (\deg F)^{n-1} .$$

Proof. Let $d_i = \deg F_i$ for $i = 1, \ldots, n$. Obviously the polynomials $F_1, \ldots, F_n$ are algebraically independent. Applying Corollary 1.4 to relations $G_i(F_1, \ldots, F_n) = X_i$ we get $\operatorname{weight} G_i \le \prod_{i=1}^{n} d_i$ and consequently

$$\deg G_i \le \frac{\operatorname{weight} G_i}{\min(d_i)} \le \frac{\prod_{i=1}^{n} d_i}{\min(d_i)} \le (\deg F)^{n-1} .$$

Example 3.2 Take $F = (X_1, X_2 + X_1^d, \ldots, X_n + X_{n-1}^d)$. Then F is a polynomial automorphism and $\deg F^{-1} = (\deg F)^{n-1}$.

4. Bézout's Inequality

Let $\mathbf{K}$ be a field (we don't assume $\mathbf{K}$ to be algebraically closed!) and let $F_1, \ldots, F_n \in \mathbf{K}[X] = \mathbf{K}[X_1, \ldots, X_n]$ and $a \in \mathbf{K}^n$. We say that a is a nondegenerate solution of the system of equations $F_i = 0$, $1 \le i \le n$ if $F_i(a) = 0$ for $i = 1, \ldots, n$ and if the determinant $\det(\partial F_i / \partial X_j)$ is nonzero at a. Using Perron's Theorem we will prove the following

Theorem 4.1 (Bézout's inequality)
Let $F_1, \ldots, F_n \in \mathbf{K}[X]$ be polynomials of degree $d_1, \ldots, d_n > 0$. Then the number of nondegenerate solutions of the system of equations $F_i = 0$, $1 \le i \le n$ is at most $\prod_{i=1}^{n} d_i$.

Proof. Replacing $\mathbf{K}$ by an infinite extension we may assume that $\mathbf{K}$ is infinite. Let $a_1, \ldots, a_s$ be nondegenerate solutions of the system of equations $F_i = 0, 1 \le i \le n$ and let $F_{n+1} = c_1X_1 + \ldots + c_nX_n$ be a linear form such that the elements $F_{n+1}(a_k)$, $k = 1, \ldots, s$ are pairwise different. Let $P = P(Y_1, \ldots, Y_{n+1}) \in \mathbf{K}[Y_1, \ldots, Y_{n+1}]$ be the minimal polynomial of $F_1, \ldots, F_{n+1}$ (note that $F_1, \ldots, F_n$ are algebraically independent since its Jacobian determinant does not vanish). Then $P = a_0(Y_1, \ldots, Y_n)Y_{n+1}^{d^*} + \ldots + a_{d^*}(Y_1, \ldots, Y_n)$ is of degree $d^* > 0$ and $d^* \le \prod_{i=1}^n d_i$ by Corollary 1.3. Let $Y = (Y_1, \ldots, Y_n)$. By the formal inverse function theorem there are formal power series $\Phi_k(Y) = (\Phi_{k,1}(Y), \ldots, \Phi_{k,n}(Y)) \in \mathbf{K}[[Y]]$, $k = 1, \ldots, s$ such that $F_i(\Phi_k(Y)) = Y_i$ for $i = 1, \ldots, n$ and $\Phi_k(0) = a_k$ for $k = 1, \ldots, s$. Let $\Psi_k(Y) = F_{n+1}(\Phi_k(Y))$ for $k = 1, \ldots, s$. Thus we get $P(Y, \Psi_k(Y)) = 0$ for $k = 1, \ldots, s$ and $\Psi_k(0) \ne \Psi_l(0)$ for $k \ne l$. Therefore $s \le d^* \le d_1 \ldots d_n$ and we are done.

Example 4.2 (Fulton[5])
Let $\mathbf{K} = \mathbf{R}$ and take $(F_1, F_2, F_3) = (\prod_{i=1}^m (X_1 - i)^2 + \prod_{j=1}^m (X_2 - j)^2, X_1X_3, X_2X_3)$. Then the polynomials F_1, F_2 and F_3 are algebraically independent and the system $F_i = 0, 1 \le i \le 3$ has m^2 solutions. On the other hand $(\deg F_1)(\deg F_2)(\deg F_3) = 2m \cdot 2 \cdot 2 = 8m$ and Bézout's inequality does not hold if $m > 8$. Thus the assumption of nondegeneracy in the above theorem is essential.

Remark 4.3 A solution $a \in \mathbf{K}^n$ of a system of equations $F_i = 0, 1 \le i \le n$ is algebraically isolated if the extension $\mathbf{K}[[X]]/\mathbf{K}[[F(a+X)]]$ is finite. Every nondegenerate solution is algebraically isolated. Using the Weierstrass Preparation Theorem we could strengthen Theorem 4.1 as follows: the number of algebraically isolated solutions of the system $F_i = 0, 1 \le i \le n$ is at most $\prod_{i=1}^n d_i$.

5. Geometric Degree

Let $\mathbf{K}$ be an infinite field. A polynomial map $F = (F_1, \ldots, F_n)$ over $\mathbf{K}$ is separable if the polynomials $F_1, \ldots, F_n$ are algebraically independent and the field extension $\mathbf{K}(X)/\mathbf{K}(F)$ is separable. We put $d(F) = (\mathbf{K}(X) : \mathbf{K}(F))$ and call $d(F)$ the geometric degree of F. Let $d_i = \deg F_i$ for $i = 1, \ldots, n$.

Proposition 5.1 *For any separable map F: $d(F) \le \prod_{i=1}^n d_i$.*

Proof. Since $\mathbf{K}(X)$ is separable over $\mathbf{K}(F)$ and $\mathbf{K}$ is infinite, we can find $c_1, \ldots, c_n \in \mathbf{K}$ such that $F_{n+1} = c_1X_1 + \ldots + c_nX_n$ is a primitive element of the extension $\mathbf{K}(X)/\mathbf{K}(F)$. Let $P = P(Y_1, \ldots, Y_{n+1})$ be the minimal polynomial of $F_1, \ldots, F_{n+1}$. Then P is a Perron's polynomial and $d(F) = \deg_{Y_{n+1}} P \le d_1 \ldots d_n$.

Here is another application of Perron's Theorem.

Proposition 5.2 *If F is separable and $d(F) > \prod_{i=1}^n d_i - \min_{i=1}^n (d_i)$ then the extension $\mathbf{K}[X]/\mathbf{K}[F]$ is integral.*

Proof. Replacing the variables $X_1, \ldots, X_n$ by their generic linear combinations we may assume that all X_i, $1 \le i \le n$ are primitive elements of the extension $\mathbf{K}(X)/\mathbf{K}(F)$. It suffices to check that any primitive element of the form $F_{n+1} = c_1X_1 + \ldots + c_nX_n$ is integral over $\mathbf{K}[F]$. Let $P = P(Y_1, \ldots, Y_{n+1}) = a_0(Y_1, \ldots, Y_n)Y_{n+1}^{d(F)} + \ldots$ be the minimal polynomial of $F_1, \ldots, F_{n+1}$. Then weight$(a_0 Y_{n+1}^{d(F)}) \leq d_1 \ldots d_n$ by Perron's Theorem and consequently

$$\deg a_0 \leq \frac{\mathrm{weight} a_0}{\min(d_i)} \leq \frac{d_1 \ldots d_n - d(F)}{\min(d_i)} < 1$$

that is a_0 is a nonzero constant. This proves the proposition.

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