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SEVERAL COMPLEX VARIABLES AND ANALYTIC SPACES

On a Theorem of Platte

by

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Summary. Given a holomorphic function $f : \mathbf{C}^n \supset U \to \mathbf{C}$ with isolated singularity we study the relation of integral dependence for f over the ring determined by its partial derivatives. We give a simple analytic proof of a theorem of E. Platte.

1. Introduction — Formulation of the theorems. Let U, V be two domains in $\mathbf{C}^n$ such that $0 \in U \cap V$. Let f be a holomorphic function in U and suppose that its gradient defines a proper holomorphic mapping

$$g = \left(\frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_n}\right) : U \to V$$

such that $g^{-1}(0) = \{0\}$. Since $g^{-1}(0) = \{0\}$ and g is proper, it is a μ-sheeted analytic covering with the critical locus $S \subset V$ and μ is the Milnor number of f at 0. Then there exists a unique monic polynomial

$$P(\cdot, T) = T^{\mu} + s_1(\cdot)T^{\mu-1} + \ldots + s_{\mu}(\cdot)$$

with coefficients holomorphic in V such that

$$P(w, T) = \prod_{z \in g^{-1}(w)} (T - f(z)) \quad \text{for } w \in V \setminus S$$

(for details see [1], Chapter 7). We have, by construction,

$$P(g(z), f(z)) = 0 \quad \text{for } z \in U.$$

The aim of this note is to present a simple analytic proof of the following theorem, due to E. Platte.

THEOREM 1 (cf. [4]). *The polynomial $P(\cdot, T)$ is irreducible in the ring $\mathcal{O}(V)[T]$ of polynomials with coefficients holomorphic in V.*

Proof is given in Section 2 of this note.

The original Platte's Theorem was given in a slight different form which, in the complex case, follows from Theorem 1 by passing to germs. Let's note that recently in [2] it has been given a proof of Platte's result supported on the trace mapping for differentials.

THEOREM 2. *Let λ be the Łojasiewicz exponent of g at 0, i.e. the smallest number Θ such that $|g(z)| \geqslant C|z|^{\Theta}$ near 0 with some positive constant C. Then*

$$\min\{1/i \operatorname{ord} s_i : i = 1, \ldots, \mu\} = 1 + 1/\lambda,$$

where ord s_i *is the degree of the initial homogeneous polynomial for s_i at 0.*

Proof. It is proved in [6] that $\min\{1/i \operatorname{ord} s_i : i = 1, \ldots, \mu\}$ is equal to the largest exponent Θ_0 for the inequality

$$|f(z)| \leqslant C |\operatorname{grad} f(z)|^{\Theta}.$$

On the other hand, in [8] there is shown that $\Theta_0 = 1+1/\lambda$, and so Theorem 2 follows.

Obviously, our theorem implies that $\operatorname{ord} s_i \geqslant i$ (even more: $\operatorname{ord} s_i \geqslant i+1$) for $i = 1, \ldots, \mu$. Thus, we get the result from [7]: the minimal polynomial of f gives rise to the integral dependence relation for f over the ideal generated by $\frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_n}$ in the local ring of holomorphic germs at 0.

2. Proof of Platte's Theorem.

To prove Theorem 1 we start with two Lemmas.

LEMMA 1. *The polynomial $P(\cdot, T)$ is irreducible in $\mathcal{O}(V)[T]$ if and only if its discriminant Δ does not vanish identically in V.*

Proof. Since the necessary condition is well known, we only prove the sufficient one. Suppose that $P(\cdot,T) = P_1(\cdot,T)P_2(\cdot,T)$ in $\mathcal{O}(V)[T]$ and choose $w \in V \setminus S$ such that $\Delta(w) \neq 0$. Since $P(g, f) = 0$ in $\mathcal{O}(U)[T]$, we may assume $P_1(g, f) = 0$. Thus, $P_1(w, f(z)) = 0$ for $z \in g^{-1}(w)$. Hence $P_1(w,T)$ has degree at least μ, because $\Delta(w) \neq 0$ implies that f has μ different values on the fibre $g^{-1}(w)$. Therefore $P_2(\cdot, T)$ is a unit of $\mathcal{O}(V)$.

LEMMA 2. *Let $\gamma = (\gamma_1, \ldots, \gamma_n) : W \to \mathbf{C}^n$ be a holomorphic mapping defined in a convex domain $W \subset \mathbf{C}^n$. If*

$$(*) \qquad w_1 \frac{\partial \gamma_1}{\partial w_k}(w) + \ldots + w_n \frac{\partial \gamma_n}{\partial w_k}(w) = 0 \quad \text{for } w \in W,\ k = 1, \ldots, n,$$

then $\gamma \mid \mathbf{R}w$ is constant for all $w \in W$.

Proof. The function $\chi = w_1\gamma_1 + \ldots + w_n\gamma_n$ is holomorphic and $(*)$ gives

$$\gamma_k = \frac{\partial \chi}{\partial w_k} \quad \text{for } k = 1, \ldots, n.$$

Thus

$$w_1 \frac{\partial \chi}{\partial w_1} + \ldots + w_n \frac{\partial \chi}{\partial w_n} = \chi$$

and differentiating gives

$$w_1 \frac{\partial^2 \chi}{\partial w_k \partial w_1} + \ldots + w_n \frac{\partial^2 \chi}{\partial w_k \partial w_n} = 0 \quad \text{for } k = 1, \ldots, n.$$

By symmetry of partial derivatives we obtain equalities

$$w_1 \frac{\partial \gamma_k}{\partial w_1} + \ldots + w_n \frac{\partial \gamma_k}{\partial w_n} = 0 \quad \text{for } k = 1, \ldots, n$$

which imply that the derivative of $t \to \gamma(wt)$ vanishes identically on its domain, and so the lemma is proved.

Proof of Theorem 1. According to Lemma 1 we have to check that the discriminant of the polynomial $P(\cdot, T)$ does not vanish identically on V. To do it, let us choose a point $b \in V\backslash\{0\}$ such that the interval $(0, b] = \{tb : t \in (0, 1]\} \subset V\backslash S$. Since b lies outside of the critical locus S, we have

(i) $g^{-1}(b) = \{a_1, \ldots, a_\mu\}$, where $a_i \neq a_j$ for $i \neq j$,

(ii) there exsists an open convex neighbourhood $W \subset V\backslash S$ of b such that $g^{-1}(w) = U_1 \cup \ldots \cup U_n$, where $U_1, \ldots, U_n$ are pairwise disjoint open connected sets and $g \mid U_i : U_i \to W$ is a biholomorphic mapping, for $i = 1, \ldots, \mu$.

Let $\varphi_i = (g \mid U_i)^{-1} : W \to U_i$. The polynomial

$$T^\mu + \sum_{i=1} (s_i \mid W) T^{\mu - i}$$

has the roots $f \circ \varphi_1, \ldots, f \circ \varphi_\mu$. If $f \circ \varphi_1, \ldots, f \circ \varphi_\mu$ are pairwise different then its discriminant does not vanish identically in W. Then to get the contradiction suppose that there exists $i \neq j$ such that

$$(1) \qquad f \circ \varphi_i = f \circ \varphi_j.$$

Differentiating (1) and making use of the relations $(\operatorname{grad} f) \circ \varphi_i = (\operatorname{grad} f) \circ \varphi_j =$ identity, we obtain

$$(2) \quad w_1 \frac{\partial(\varphi_{i1} - \varphi_{j1})}{\partial w_k} + \ldots + w_n \frac{\partial(\varphi_{in} - \varphi_{jn})}{\partial w_k} = 0 \quad \text{for } w \in W,\ k = 1, \ldots, n$$

Conditions (2), by Lemma 2, imply that the mapping

$$\varphi_i - \varphi_j = (\varphi_{i1} - \varphi_{j1}, \ldots, \varphi_{in} - \varphi_{jn})$$

is constant on every vector line through origin. In particular, we obtain

(3) $\varphi_i(w) - \varphi_j(w) = \varphi_i(b) - \varphi_j(b) = c \neq 0$ for $w \in W \cap (0, b]$.

Since $g^{-1}(0) = \{0\}$, ther exists $b' \in (0, b)$ such that the fiber $g^{-1}(b')$ lies in the ball $|z| \leqslant \frac{1}{4}|c|$.

Let $\alpha(t) = tb' + (1-t)b$ for $t \in [0,1]$. Since $\alpha(t)$ lies outside of the critical locus S, there exist the unique lifted analytic arcs $\alpha_l : [0,1] \to U \backslash g^{-1}(S)$ such that $g \circ \alpha_l = \alpha$ and $\alpha_l(0) = a_l$ for $l = 1, \ldots, \mu$. By (3) we have

$$\alpha_i(t) - \alpha_j(t) = c \quad \text{for } t \in [0,1] \text{ and } |c| = |\alpha_i(1) - \alpha_j(1)| \leqslant 1/2|c|$$

contradicts $c \neq 0$. Consequently (1) does not hold and Theorem 1 is proved.

3. Observations and another statement of Platte's theorem. In order to put the irreducibility of the polynomial $P(\cdot, T)$ in a more general context, suppose that U is an irreducible analytic subset of $\mathbb{C}^N$ of dimension n and that, as before, V is a domain in $\mathbb{C}^n$. For a given μ-sheeted analytic covering $g : U \to V$ and $f \in \mathcal{O}(U)$ the polynomial $P(\cdot, T)$ can be constructed by the same way as in Section 1. The mapping

$$g^* : \mathcal{O}(V) \ni h \to h \circ g \in \mathcal{O}(U)$$

is an injective homomorphism of two domains. Let $L = \mathrm{Fr}\mathcal{O}(U)$ (the field of fractions of $\mathcal{O}(U)$) and let $K = \mathrm{Fr}(g^*(\mathcal{O}(V)))$. Since the ring $g^*(\mathcal{O}(V))$ (isomorphic to $\mathcal{O}(V)$) is normal, the monic polynomial

$$P^0(u, T) = T^\mu + s_1(g(u))T^{\mu-1} + \ldots + s_\mu(g(u))$$

is irreducible in $g^*(\mathcal{O}(V))[T]$ if and only if it is irreducible in $K[T]$. Then, as in Lemma 1, and by a well-known field extensions properties (cf. [9], Chapter 6, p. 107) one gets

PROPOSITION 1. *The following four conditions are equivalent*

(a) *The polynomial $P(\cdot, T)$ is irreducible in $\mathcal{O}(V)[T]$.*

(b) *The discriminant of $P(\cdot, T)$ does not vanish identically on V.*

(c) *There exists a $w \in V$ such that f has μ different values on the fibre $g^{-1}(w)$.*

(d) *f is a primitive element of the field extension $K \subset L$.*

According to Proposition 1, Theorem 1 can be stated as follows: *the function f is a primitive element of the field extension* $\mathrm{Fr}(\mathrm{grad}\, f)^*\mathcal{O}(V) \subset \mathrm{Fr}\,\mathcal{O}(U)$.

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