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SEVERAL COMPLEX VARIABLES AND ANALYTIC SPACES

Remarks on Polynomial Equations with Analytic Coefficients

by

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Summary. We prove Hilbert's Nullstellensatz and Max Noether's Theorem for polynomial equations with coefficients in the ring of convergent power series.

1. Nullstellensatz for polynomial equations with analytic coefficients. In this note we shall consider polynomials with coefficients in the ring $\mathbf{C}\{X\}$ of convergent power series in variables $X = (X_1, \ldots, X_m)$. Let $F(X, Y) = (F_1(X, Y), \ldots, F_N(X, Y)) \in \mathbf{C}\{X\}[Y]^N$ be a sequence of polynomials in variables $Y = (Y_1, \ldots, Y_n)$. We denote by $F(X, Y)\mathbf{C}\{X\}[Y]$ resp. $F(X, Y)\mathbf{C}\{X, Y\}$ the ideal generated by $F_1(X, Y), \ldots, F_N(X, Y)$ in the ring $\mathbf{C}\{X\}[Y]$ resp. in the ring $\mathbf{C}\{X, Y\}$ and $\mathbf{C}\{X\}[Y]/F(X, Y)$ resp. $\mathbf{C}\{X, Y\}/F(X, Y)$ the corresponding quotient algebras. In what follows, we denote by U an open neighbourhood (nbhd) of the origin $O \in \mathbf{C}^m$ contained in the domains of convergence of all coefficients of the polynomials $F(X, Y)$, $G(X, Y)$.

THEOREM (1.1). *Suppose that there exists an open nbhd U of the origin $O \in \mathbf{C}^m$ such that for each $(x, y) \in U \times \mathbf{C}^m$ from $F(x, y) = 0$ follows $G(x, y) = 0$. Then there is an integer $q > 0$ such that $G(X, Y)^q \in F(X, Y)\mathbf{C}\{X\}[Y]$.*

Proof. We shall get (1.1) from the local version of Hilbert's Nullstellensatz (cf. [5] where a simplified proof of this theorem is given). Let us assume first that $F_1(X, Y), \ldots, F_N(X, Y)$ and $G(X, Y)$ are homogeneous polynomials in variables Y. By the local Nullstellensatz we have $G(X, Y)^q = \sum_{l=1}^{N} Q_l(X, Y) F_l(X, Y)$ with $Q_l(X, Y) \in \mathbf{C}\{X, Y\}$.

Write $Q_l(X, Y) = \sum_k Q_l^{(k)}(X, Y)$ where $Q_l^{(k)}$ is a homogeneous form in Y of degree k or $Q_l^{(k)} = 0$. Comparing terms of the same degree, we conclude that $G(X, Y) = \sum_{(k,l)\in I} Q_l^{(k)}(X, Y) F_l(X, Y)$ where

$$I = \{(k, l)\colon \deg Q_l^{(k)} + \deg F_l = q \deg G\},$$

so $G(X, Y)^q \in F(X, Y)\mathbf{C}\{X\}[Y]$. Let us consider now the general case. Let Y_0 be a new variable, $\tilde{Y} = (Y_0, Y)$ and let $\tilde{P}(X, \tilde{Y}) = \sum_k Y_0^{\deg P - k} P^{(k)}(X, Y)$ for any polynomial $P(X, Y) = \sum_k P^{(k)}(X, Y) \in \mathbf{C}\{X\}[Y]$. Then $\tilde{P}(X, \tilde{Y})$ is a homogeneous form in $\tilde{Y}$ of degree $\deg P$ and $\tilde{P}(X, 1, Y) = P(X, Y)$. From the assumption it follows that $\tilde{F}(x, \tilde{y}) = 0$, $(x, \tilde{y}) \in U \times \mathbf{C}^{n+1}$ implies $y_0 \tilde{G}(x, \tilde{y}) = 0$. Hence by the first part of the proof we get $[Y_0 \tilde{G}(X, \tilde{Y})]^q = \sum_{l=1}^{N} P_l(X, \tilde{Y}) \tilde{F}_l(X, \tilde{Y})$ with $P_l(X, \tilde{Y}) \in \mathbf{C}\{X\}[\tilde{Y}]$. Putting $Y_0 = 1$ we get the theorem. Let us note the following corollary of (1.1).

Corollary (1.2). *Suppose that there exist an open nbhd U of the origin $O \in \mathbf{C}^m$ such that the system of equations $F(X, Y) = 0$ has no solutions in $U \times \mathbf{C}^n$. Then $1 \in F(X, Y)\mathbf{C}\{X\}[Y]$.*

In the sequel the following will be useful.

Property (1.3). $\mathbf{C}\{X\}[Y]/F(X, Y)$ is a finite $\mathbf{C}\{X\}$-module if and only if for any $j = 1, \ldots, n$ the ideal $F(X, Y)\mathbf{C}\{X\}[Y]$ contains a monic polynomial $H_j(X, Y_j) \in \mathbf{C}\{X\}[Y]$.

The above property follows easily from the well-known properties of integral extensions. Let us recall that a continuous mapping is finite if it is closed and has finite fibers. The basic properties of finite mappings are given in [3]. To abbreviate the notation let us put $N_U(F) = \{(x, y) \in U \times \mathbf{C}^n : F(x, y) = 0\}$. We claim

Proposition (1.4). *The following two conditions are equivalent:*

(i) There is a basis of open nbhds U such that $N_U(F) \neq 0$ and the projection $pr_1 : N_U(F) \to U$ given by $pr_1(x, y) = x$ is finite.

(ii) $\mathbf{C}\{X\}[Y]/F(X, Y)$ is a non-zero finite $\mathbf{C}\{X\}$-module.

Proof. Assume (ii). By (1.3) the ideal $F(X, Y)\mathbf{C}\{X\}[Y]$ contains monic polynomials $H_j(X, Y_j)$ $(j = 1, \ldots, n)$. Let $H(X, Y) = \big(H_1(X, Y_1), \ldots, H_n(X, Y_n)\big)$.

We have $H(X, Y)\mathbf{C}\{X\}[Y] \subset F(X, Y)\mathbf{C}\{X\}[Y]$, so $N_U(F) \subset N_U(H)$ for sufficiently small open nbhd U. $N_U(F)$ is a closed and by (1.2) non-empty subset of $N_U(H)$, the projection $N_U(H) \ni (x, y) \to x \in U$ is finite (cf. [3], p. 51). Hence $pr_1 : N_U(F) \to U$ is finite, which proves (i). Let us suppose now that (i) holds. Using some basic properties of finite mappings (cf. [3], p. 48) we confirm that (a) $N_0 = \{y \in \mathbf{C}^n\colon F(0, y) = 0\}$ is a non-empty finite set, (b) for $x \in \mathbf{C}^m$ sufficiently near 0 the solutions of the system of equations $F(x, Y) = 0$ lie in a given nbhd of the set N_0. It follows that for any $y \in N_0$ the system of equations $X = 0$, $F(X, Y + y) = 0$ has near $0 \in \mathbf{C}^{m+n}$ only the solution $X = 0$, $Y = 0$, so by the local Nullstellensatz the homomorphism $\mathbf{C}\{X\} \to \mathbf{C}\{X, Y\}/F(X, Y + y)$ is quasi-finite. Consequently it is finite (cf. [2], p. 88) and the ideal $F(X, Y + y)$ $\mathbf{C}\{X, Y\}$ contains distinguished polynomials $H_j^{(y)}(X, Y_j)$ $(j = 1, \ldots, n)$ (cf. [2],

p. 93). Let $H_j(X, Y_j) = \prod_{y \in N_0} H_j^{(y)}(X, Y_j - y_j)$. Using the properties (a) and (b) we check that the polynomials $H_j(X, Y_j)$ vanish on $N_U(F)$ provided the nbhd U is sufficiently small, so by (1.1) there is an integer $q > 0$ such that $H_j(X, Y_j)^q \in F(X, Y)\mathbf{C}\{X\}[Y]$ $(j = 1, \ldots, n)$. According to (1.3) this implies (i).

2. Euclidean and Weierstrass Division. Let $H(Y) = (H_1(Y_1), \ldots, H_n(Y_n)) \in A[Y]^n$ be a sequence of monic polynomials with coefficients in a commutative ring A with identity. Let us quote the following two well-known results (cf. [1], [3], [4]).

THEOREM (2.1) (Generalized Euclidean Division). *For any $G(Y) \in A[Y]$ there is a unique $G^*(Y) \in A[Y]$ such that $G(Y) \equiv G^*(Y) \bmod H(Y)A[Y]$ and $\deg_{Y_j} G^* < \deg_{Y_j} H$ for $j = 1, \ldots, n$.*

THEOREM (2.2) (Generalized Weierstrass Division). *Let $H(X, Y) = (H_1(X, Y_1), \ldots, H_n(X, Y_n)) \in \mathbf{C}\{X, Y\}$ be a sequence of power series such that $H_j(O, Y_j) \neq 0$ in $\mathbf{C}\{Y_j\}$ for $j = 1, \ldots, n$. For any $G(X, Y) \in \mathbf{C}\{X, Y\}$ there is a unique $G_*(X, Y) \in \mathbf{C}\{X\}[Y]$ such that $G \equiv G_* \bmod H(X, Y)\mathbf{C}\{X, Y\}$ and $\deg_{Y_j} G_* < \mathrm{ord}_{Y_j} H_j(O, Y_j)$ for $j = 1, \ldots, n$.*

Suppose now that $H(X, Y)$ is a sequence of distinguished polynomials, so $\mathrm{ord}_{Y_j} H(O, Y_j) = \deg_{Y_j} H(X, Y_j)$ $(j = 1, \ldots, n)$. Then, by the uniqueness part of (2.2) we have for any $G(X, Y) \in \mathbf{C}\{X\}[Y]$: $G^*(X, Y) = G_*(X, Y)$. In particular we get

COROLLARY (2.3) (cf. [1] p. 206). *If $H(X, Y) = (H_1(X, Y_1), \ldots, H_n(X, Y_n)) \in \mathbf{C}\{X\}[Y]^n$ is a sequence of distinguished polynomials then $H(X, Y)\mathbf{C}\{X, Y\} \cap \mathbf{C}\{X\}[Y] = H(X, Y)\mathbf{C}\{X\}[Y]$.*

The aim of this section is

THEOREM (2.4). *Let $H(X, Y) = (H_1(X, Y), \ldots, H_n(X, Y_n)) \in \mathbf{C}\{X\}[Y]^n$ be* a sequence of monic polynomials. Let $m_j(y) = \mathrm{ord}_{Y_j} H_j(O, Y_j + y_j)$ $(j = 1, \ldots, n)$ for $y \in \mathbf{C}^n$. Then for any family $\{R^{(y)}(X, Y)\}$ of elements of the ring $\mathbf{C}\{X\}[Y]$ such that $\deg_{Y_j} R^{(y)}(X, Y) < m_j(y) (j = 1, \ldots, n)$ for $y \in \mathbf{C}^n$ there is a unique $R(X, Y) \in \mathbf{C}\{X\}[Y]$ such that $R(X, Y+y) \equiv R^{(y)}(X, Y) \bmod H(X, Y+y)\mathbf{C}\{X, Y\}$ and $\deg_{Y_j} R(X, Y) < \deg_{Y_j} H_d(X, Y_j) (j = 1, \ldots, n)$.

Our proof of (2.4) is based on Lemma (2.6) presented below. Retaining the notation adopted in the beginning of this section assume that the polynomial $H_j(Y_j)$ has a factorisation into pairwise strictly comprime monic polynomials $H_j^{(y_j)}(Y_j) (y_j \in I_j)$ i.e. $H_j(Y_j) = \prod_{y_j \in I_j} H_j^{(y_j)}(Y_j)$ with $1 \in (H_j^{(y_j)}(Y_j), H_j^{(\bar{y}_j)}(Y_j))A[Y_j]$ provided $y_j \neq \bar{y}_j$. Here I_j is a non-empty set which may be infinite, we assume $H_j^{(y_j)}(Y_j) = 1$ for all but finite number of $y_j \in I_j$. Let $H^{(y)}(Y) = (H_1^{(y_1)}(Y_1), \ldots, H_n^{(y_n)}(Y_n))$ for any $y = (y_1, \ldots, y_n) \in I = I_1 \times \ldots \times I_n$. The ideals $H^{(y)}(Y)A[Y]$, $y \in I$ are pairwise comaximal, so for $n = 1$ (one variable Y) we have $\bigcap_{y \in I} H^{(y)}(Y)A[Y] = \prod_{y \in I} H^{(y)}(Y)A[Y] = H(Y)A[Y]$. By induction on the number n of variables Y we check

LEMMA (2.5). $\bigcap_{y\in I} H^{(y)}(Y)A[Y] = H(Y)A[Y]$.

Proof. Let $n > 1$ and suppose that (2.5) holds for $n-1$. Fix $(y_1, \ldots, y_{n-1}) \in I_1 \times \ldots \times I_{n-1}$ and let $\bar{A} = A[Y_1, \ldots, Y_{n-1}]/(H_1^{(y_1)}(Y_1), \ldots, H_{n-1}^{(y_{n-1})}(Y_{n-1})) = A[\bar{Y}_1, \ldots, \bar{Y}_{n-1}]$ where $\bar{Y}_j$ is the residue class of Y_j. Let $G(Y) \in \bigcap_{y\in I} H^{(y)}(Y)A[Y]$. Then $G(\bar{Y}_1, \ldots, \bar{Y}_{n-1}, Y_n) \in H_n^{(y_n)}(Y_n)\bar{A}[Y_n]$ for all $y_n \in I_n$, so by the case $n = 1$ we get $G(\bar{Y}_1, \ldots, \bar{Y}_{n-1}, Y_n) \in H_n(Y_n)\bar{A}[Y_n]$. Hence there is a polynomial $Q_n(Y) \in A[Y]$ such that $G(Y) = H_n(Y_n)\cdot Q_n(Y) \mathrm{mod}(H_1^{(y_1)}(Y_1), \ldots, H_{n-1}^{(y_{n-1})}(Y_{n-1}))$. Applying the induction hypothesis we get $G(Y) \in H(Y)A[Y]$. This proves the lemma since the inclusion $H(Y)A[Y] \subset \bigcap_{y\in I} H^{(y)}(Y)A[Y]$ is obvious.

LEMMA (2.6). *For any family $\{r^{(y)}(Y)\}_{y\in I}$ of polynomials from $A[Y]$ such that $deg_{Y_j} r^{(y)}(Y) < deg_{Y_j} H^{(y_j)}(Y_j)$ $(j = 1, \ldots, n)$ for all $y \in I$ there is a unique $r(Y) \in A[Y]$ such that $r(Y) \equiv r^{(y)}(Y) mod H^{(y)}(Y)A[Y]$ for $y \in I$ and* $\deg_{Y_j} r(Y) < deg_{Y_j} H_j(Y_j)$ $(j = 1, \ldots, n)$.

Proof. By the Chinese Remainder Theorem there is a $G(Y) \in A[Y]$ such that $G(Y) \equiv r^{(y)}(Y) \mathrm{mod} H^{(y)}(Y)A[Y]$. Let $G^*(Y)$ be such that in (2.1).

One sees easily that we may take $r(Y) = G^*(Y)$. The uniqueness follows from (2.5) and from the uniqueness of the remainder in (2.1). Now, we are in a position to prove theorem (2.4).

Proof of (2.4). By Hensel's lemma we may write $H_j(X, Y_j) = \prod_{y_j\in \mathbf{C}} H_j^{(y_j)}(X, Y_j)$ with $H_j^{(y_j)}(O, Y_j) = (Y_j - y_j)^{m_j(y)}$. Let $H^{(y)}(X, Y) = (H_1^{(y_1)}(X, Y_1), \ldots, H_n^{(y_n)}(X, Y_n))$ for any $y = (y_1, \ldots, y_n) \in \mathbf{C}^n$. Let us note the following two properties:

a) $H(X, Y+y)\mathbf{C}\{X, Y\} = H^{(y)}(X, Y+y)\mathbf{C}\{X, Y\}$ by Hensel's lemma

b) $H^{(y)}(X, Y+y)\mathbf{C}\{X, Y\} \cap \mathbf{C}\{X\}[Y] = H^{(y)}(X, Y+y)\mathbf{C}\{X\}[Y]$ by Corollary (2.3).

Let $R(X, Y) \in \mathbf{C}\{X\}[Y]$. Using a) and b) we check that the congruences $R(X, Y+y) \equiv R^{(y)}(X, Y) \mathrm{mod} H(X, Y+y)\mathbf{C}\{X, Y\}$ and $R(X, Y) \equiv R^{(y)}(X, Y-y) \mathrm{mod} H^{(y)}(X, Y)\mathbf{C}\{X\}[Y]$ are equivalent. Now (2.4) follows from (2.6) by taking in (2.6) $r^{(y)}(X, Y) = R^{(y)}(X, Y-y)$.

3. Max Noether's Theorem. Theorem (3.1) given below can be viewed as a generalisation of Max Noether's $af+bg$ theorem.

THEOREM (3.1). *Suppose that $F(X, Y) \in \mathbf{C}\{X\}[Y]^N$ satisfies the equivalent conditions of Proposition* (1.4). *Let $G(X, Y) \in \mathbf{C}\{X\}[Y]$ be such that $G(X, Y+y) \in F(X, Y+y)\mathbf{C}\{X, Y\}$ for all $y \in \mathbf{C}^n$. Then $G(X, Y) \in F(X, Y)\mathbf{C}\{X\}[Y]$.*

Before giving the proof of (3.1) let us remark that the local conditions $G(X, Y+y) \in F(X, Y+y)\mathbf{C}\{X, Y\}$ are relevant only for finite number of $y \in \mathbf{C}^n$ such that $F(O, y) = 0$.

Indeed $F(X, Y+y)\mathbf{C}\{X, Y\} \neq \mathbf{C}\{X, Y\}$ if and only if $F(O, y) = 0$.

Proof of (3.1). We prove first the theorem for the sequence of monic polynomials $H(X, Y) = (H_1(X, Y_1), \ldots, H_n(X, Y_n)) \in \mathbf{C}\{X\}[Y]^n$. Let $G(X, Y) \in \mathbf{C}\{X\}[Y]$ be such that $G(X, Y+y) \equiv 0 \bmod H(X, Y+y)\mathbf{C}\{X, Y\}$, $y \in \mathbf{C}^n$. Letting $G^*(X, Y)$ such that in (2.1) we check easily that $G^*(X, Y+y) \equiv 0 \bmod H(X, Y+y)\mathbf{C}\{X, Y\}$, $y \in \mathbf{C}^n$, moreover we have $\deg_{Y_j} G^* < \deg_{Y_j} H_j(X, Y_j)$ $(j = 1, \ldots, n)$. According to (2.4) this relationships imply $G^*(X, Y) = 0$, so $G(X, Y) \in H(X, Y)\mathbf{C}\{X\}[Y]$.

Let us now pass to the general case. By Property (1.3) there is a sequence of monic polynomials $H(X, Y) = (H_1(X, Y_1), \ldots, H_n(X, Y_n))$ such that $H_j(X, Y_j) \in F(X, Y)\mathbf{C}\{X\}[Y]$ $(j = 1, \ldots, n)$, so $H(X, Y+y)\mathbf{C}\{X, Y\} \subset F(X, Y+y)\mathbf{C}\{X, Y\}$ for all $y \in \mathbf{C}^n$. Let $m_j(y) = \operatorname{ord}_{Y_j} H_j(O, Y_j + y_j)$. By assumption we may write $G(X, Y+y) = \sum_{l=1}^{N} Q_l^{(y)}(X, Y)F_l(X, Y+y)$ with $Q_l(X, Y) \in \mathbf{C}\{X, Y\}$. Using (2.2) we get polynomials $R_l^{(y)}(X, Y) \in \mathbf{C}\{X\}[Y]$ such that $\deg_{Y_j} R_l^{(y)}(X, Y) < m_j(y)$ $(j = 1, \ldots, n)$ and $Q_l^{(y)}(X, Y) \equiv R_l^{(y)}(X, Y) \bmod H(X, Y+y)\mathbf{C}\{X, Y\}$. Hence we get $G(X, Y+y) \equiv \sum_{l=1}^{N} R_l^{(y)}(X, Y)F(X, Y+y) \bmod H(X, Y+y)\mathbf{C}\{X, Y\}$. According to (2.4) there are polynomials $R_l(X, Y) \in \mathbf{C}\{X\}[Y]$ such that $\deg_{Y_j} R_l(X, Y) < \deg_{Y_j} H(X, Y_j)$ $(j = 1, \ldots, n)$ and $R_l(X, Y+y) \equiv R_l^{(y)}(X, Y) \bmod H(X, Y+y)\mathbf{C}\{X, Y\}$. So we may write $G(X, Y+y) \equiv \sum_{l=1}^{N} R_l(X, Y+y)F_l(X, Y+y) \bmod H(X, Y+y)\mathbf{C}\{X, Y\}$ and by the first part of the proof we get $G(X, Y) \equiv \sum_{l=1}^{N} R_l(X, Y)F_l(X, Y) \bmod H(X, Y)\mathbf{C}\{X\}[Y]$. Hence $G(X, Y) \equiv 0 \bmod F(X, Y)\mathbf{C}\{X\}[Y]$ since $H(X, Y)\mathbf{C}\{X\}[Y] \subset F(X, Y)\mathbf{C}\{X\}[Y]$.

Finally we note a generalisation of Hensel's lemma (cf. [4], Th. 23.11).

THEOREM (3.2). *Suppose that* $F(X, Y) \in \mathbf{C}\{X\}[Y]^N$ *satisfies the equivalent conditions of Propositions* (1.4). *Then the injections* $\mathbf{C}\{X\}[Y] \ni P(X, Y) \to P(X, Y+y) \in \mathbf{C}\{X, Y\}$ *induce an isomorphism*

$$\mathbf{C}\{X\}[Y]/F(X, Y) \to \prod_{y \in \mathbf{C}^n} C\{X, Y\}/F(X, Y+y).$$

Proof. Let Φ_y: $\mathbf{C}\{X\}[Y] \to \mathbf{C}\{X, Y\}/F(X, Y+y)$ be the composite of the injection $\mathbf{C}\{X[Y] \ni P(X, Y) \to P(X, Y+y) \in \mathbf{C}\{X, Y\}$ and the natural homomorphism

$$\mathbf{C}\{X, Y\} \to \mathbf{C}\{X, Y\}/F(X, Y+y).$$

It suffices to check the following properties

a) Φ_y is surjective

b) The ideals $\ker \Phi_y$ are pairwise comaximal

c) $\bigcap_{y \in \mathbf{C}^n} \ker \Phi_y = F(X, Y)\mathbf{C}\{X\}[Y]$.

Let $H(X, Y) = (H_1(X, Y_1), \ldots, H_n(X, Y_n))$ be a sequence of monic polynomials such that $H(X, Y)\mathbf{C}\{X\}[Y] \subset F(X, Y)\mathbf{C}\{X\}[Y]$. One gets easily a) from the surjectivity of the homomorphism $\mathbf{C}\{X\}[Y] \to \mathbf{C}\{X, Y\}/H(X, Y+y)$. Property b) follows from the fact that the ideals $H(X, Y+y)\mathbf{C}\{X, Y\}$ are pairwise comaximal. Property c) we get from (3.1): $\bigcap_{y \in \mathbf{C}^n} \ker \Phi_y = \bigcap_{y \in \mathbf{C}^n} (F(X, Y+y)\mathbf{C}\{X, Y\} \cap \mathbf{C}\{X\}[Y]) = F(X, Y)\mathbf{C}\{X\}[Y]$.

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А. Плоски, **Замечания о полиномиальных уравнениях с аналитическими коэффициентами**

Доказывается теорема Гильберта о корнях и теорема Нётера для полиномиальных уравнений с коэффициентами в кольце степенных рядов.