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An approach to plane algebroid branches

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Abstract We consider a new approach to the local geometry of plane algebraic curves that allows us to obtain the basic results of the theory of plane algebroid branches over algebraically closed fields of arbitrary characteristic. We do not use the Hamburger-Noether expansions. Our basic tool is the logarithmic distance on the set of branches satisfying the strong triangle inequality which permits to make calculations directly on the equations of branches.

Keywords Plane algebroid curve · Branch · Semigroup associated with a branch · Key polynomials · Logarithmic distance · Abhyankar-Moh theory

Mathematics Subject Classification Primary 32S05; Secondary 14H20

What can be explained on fewer principles is explained needlessly by more. William of Ockham (1280–1349)[1]

[1] Quoted after Samuel E. Stumpf. Socrates to Sartre. A History of Philosophy. Mc Graw-Hill, Inc. 1993.

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1 Introduction

We present a new approach to the theory of plane algebroid branches over an algebraically closed field of arbitrary characteristic. We prove many of the classical results which can be interpreted in terms of the semigroup associated with a branch. The presented results are well-known but our proofs are new. In contrast to classical treatments of the subject given by Ancochea [6], Lejeune-Jalabert [29], Moh [32], Angermüller [7], Russell [38] and Campillo [11] we do not use the quadratic transformations. To avoid the Hamburger-Noether expansions we base our approach on the direct construction of key polynomials (the notion introduced by MacLane [30] in 1936) given by Seidenberg in his PhD thesis on the valuation ideals in polynomial rings. As far as we know the Seidenberg article of [40] is the first publication in which appears the God-given inequality $n_k\overline{\beta_k} < \overline{\beta_{k+1}}$ (we use the notation introduced by Zariski).

In all this paper we use the strong triangle inequality (STI) proved by the second author in 1985. It allows to give simple proofs of the Abhyankar–Moh theory, the Bresinsky–Angermüller characterization of semigroups associated with branches and the description of branches with given semigroup.

A plane algebroid branch may be given either by an irreducible equation $f(x, y) = 0$ or by a parametrization $x = \phi(t)$, $y = \psi(t)$. The treatments of the subject which use the Hambuger-Noether expansions (or Puiseux' expansions in the case of characteristic 0) are based on the interplay between the equations and the parametrizations of branches. In this paper after having proved the STI we make calculations on the equations of branches without recourse to their parametrizations. In this way we get shorter and conceptually simpler proofs of basic theorems than in the classical approach to plane algebroid branches.

In this paper we do not pretend to completeness. For the embedded resolution process by blowing-ups as well as topological properties of singularities we refer the reader to the book of C.T.C.Wall [46]. The approach based on the classical notion of infinitely near points is presented in the book by Casas-Alvero [13], where many more aspects of plane curves singularities are treated.

The paper is organized as follows. We review in Sect. 2 the results on plane algebroid curves needed in the sequel. In particular, we prove the strong triangle inequality. In Sect. 3 we study the structure of the semigroup associated with a plane branch following the method of Seidenberg. Section 4 is devoted to the concept of key polynomial and to the Abhyankar–Moh theory of approximate roots. In Sect. 5 we prove a version of the Abhyankar–Moh irreducibility criterion which is the basic tool for studying the branches with given semigroup (Sects. 6, 7). We finish Sect. 7 with Abhyankar's irreducibility criterion in terms of generalized Newton's diagrams.

The following notation is used in the sequel. The set of all integers (resp. nonnegative integers) is denoted by **Z** (resp. **N**). We write gcd S for the greatest common divisor of a nonempty subset $S \subset \mathbf{N}$. Conventions about calculating with $+\infty$ are usual. In all this note **K** is an algebraically closed field of arbitrary characteristic.

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2 Preliminaries

In this section we fix our notations and recall some useful notions and results.

2.1 Arithmetical lemmas and semigroups of naturals

We state here without proof some properties of semigroups of natural numbers that we will use in Sect. 3 of this paper. For the proofs we refer the reader to [7] (see also [26]).

Proposition 2.1 *Let $v_0, \dots, v_k$ be a sequence of positive integers. Set $d_i = \gcd(v_0, \dots, v_i)$ for $i \in \{0, 1, \dots, k\}$ and $n_i = \frac{d_{i-1}}{d_i}$ for $i \in \{1, \dots, k\}$. Then for every $a \in \mathbf{Z}d_k$ we have Bézout's relation:*

$$a = a_0 v_0 + a_1 v_1 + \cdots + a_k v_k,$$

where $a_0 \in \mathbf{Z}$ and $0 \le a_i < n_i$ for $i \in \{1, \dots, k\}$. The sequence $(a_0, \dots, a_k)$ is unique. Assume that $n_{i-1} v_{i-1} \le v_i$ for $i \in \{2, \dots, k\}$. Then for $a = n_k v_k$ we have $a_0 > 0$. If $a \in \mathbf{N}v_0 + \cdots + \mathbf{N}v_k$ then $a_0 \ge 0$.

Remark 2.2 Suppose that $v_k \notin \mathbf{N}v_0 + \cdots + \mathbf{N}v_{k-1}$. Then $n_k > 1$ and $n_{k-1} v_{k-1} < v_k$.

Let $n > 0$ be an integer. A sequence of positive integers $(v_0, \dots, v_h)$ is said to be a *Seidenberg n-characteristic sequence* or *n-characteristic sequence* if $v_0 = n$ and it satisfies the following two axioms

1. Set $d_i = \gcd(v_0, \dots, v_i)$ for $0 \le i \le h$ and $n_i = \frac{d_{i-1}}{d_i}$ for $1 \le i \le h$. Then $d_h = 1$ and $n_i > 1$ for $1 \le i \le h$.
2. $n_{i-1} v_{i-1} < v_i$ for $2 \le i \le h$.

Note that condition (2) implies that the sequence $(v_1, \dots, v_h)$ is strictly increasing. If $n > 1$ then $h \ge 1$. If $h = 1$ then the sequence (v_0, v_1) is a Seidenberg n-characteristic sequence if and only if $v_0 = n$ and $\gcd(v_0, v_1) = 1$. There is exactly one 1-sequence which is (1). Note also that $2^h \le n$.

If $(v_0, \dots, v_h)$ is an n-characteristic sequence then for any $k \in \{1, \dots, h\}$ the sequence $\left(\frac{v_0}{d_k}, \dots, \frac{v_k}{d_k}\right)$ is an $\frac{n}{d_k}$-characteristic sequence. Its associated sequences are $\left(\frac{d_0}{d_k}, \dots, \frac{d_k}{d_k}\right)$ and $(n_1, \dots, n_k)$.

We say that a subset G of $\mathbf{N}$ is a *semigroup* if it contains 0 and if it is closed under addition.

Let G be a nonzero semigroup and let $n \in G$, $n > 0$. Then there exists (cf. [26], Chapter 6, Proposition 6.1) a unique sequence $v_0, \dots, v_h$ such that $v_0 = n$, $v_k = \min(G \backslash \mathbf{N}v_0 + \cdots + \mathbf{N}v_{k-1})$ for $k \in \{1, \dots, h\}$ and $G = \mathbf{N}v_0 + \cdots + \mathbf{N}v_h$. We call the sequence $(v_0, \dots, v_h)$ the *n-minimal system of generators of* G. If $n = \min(G \backslash \{0\})$ then we say that $(v_0, \dots, v_h)$ is the *minimal set* of generators of G. Clearly $\gcd G = \gcd(v_0, \dots, v_h) = d_h$. If $\gcd G = 1$ then G is said to be a *numerical semigroup*.

Proposition 2.3 *Let G be a numerical semigroup with n-minimal system of generators $(v_0, \dots, v_h)$. Suppose that $n_{i-1} v_{i-1} \le v_i$. Then*

1. *The sequence* $(v_0, \dots, v_h)$ *is an* n*-characteristic sequence.*
2. $\min(G\backslash\{0\}) = \min(v_0, v_1)$.
3. *The minimal system of generators of* G *is* $(v_0, v_1, \dots, v_h)$ *if* $v_0 < v_1$, $(v_1, v_0, \dots, v_h)$ *if* $v_1 < v_0$ *and* $v_0 \not\equiv 0$ *(mod* v_1*) and* $(v_1, v_2, \dots, v_h)$ *if* $v_0 \equiv 0$ *(mod* v_1*). Moreover, the minimal system of generators of* G *is a* $\min(G\backslash\{0\})$*-characteristic sequence.*
4. *Let* $c = \sum_{k=1}^{h}(n_k - 1)v_k - v_0 + 1$. *Then for every* $a, b \in \mathbf{Z}$*: if* $a + b = c - 1$ *then exactly one element of the pair* (a, b) *belongs to* G*. Consequently* c *is the smallest element of* G *such that all integers bigger than or equal to it are in* G*.*
5. c *is an even number and* $\sharp(\mathbf{N}\backslash G) = \frac{c}{2}$.
6. *Each* v_k, $k > 0$ *is an irreducible element of* G*, that is* v_k *is not a sum of two nonzero elements of the semigroup* G*.*

The number c is called the *conductor* of the semigroup G. To check the properties of the conductor quoted in Proposition 2.3 one can adopt the elegant proof of the Conductor Formula for the planar semigroups given in [39].

2.2 Plane algebroid curves

We review here some basic notions from the local theory of algebraic curves. For more details we refer the reader to [41].

Let $f \in \mathbf{K}[[x, y]]$ be a non-zero power series without constant term. An algebroid curve $\{f = 0\}$ is defined to be the ideal generated by f in $\mathbf{K}[[x, y]]$. We say that $\{f = 0\}$ is irreducible (reduced) if f in $\mathbf{K}[[x, y]]$ is irreducible (f has no multiple factors). The irreducible curves are also called *branches*. The order $\operatorname{ord} f$ of the power series f is, by definition, the *multiplicity* of the curve $\{f = 0\}$. The *initial form* $\operatorname{in} f$ of f defines the tangent lines of $\{f = 0\}$. If $\{f = 0\}$ is irreducible then it has only one tangent line i.e. $\operatorname{in} f = l^{\operatorname{ord} f}$ where l is a linear form.

A *formal isomorphism* Φ is a pair of power series $\Phi(x, y) = (ax + by + \cdots, a'x + b'y + \cdots)$ where $ab' - a'b \neq 0$ and the dots denote terms in x, y of order bigger than 1. The map $f \longrightarrow f \circ \Phi$ is an isomorphism of the ring $\mathbf{K}[[x, y]]$. Two curves $\{f = 0\}$ and $\{g = 0\}$ are said to *be formally equivalent* if there is a formal isomorphism Φ such that $f \circ \Phi = g \cdot \text{unit}$.

For any power series $f, g \in \mathbf{K}[[x, y]]$ we define the *intersection multiplicity or intersection number* $i_0(f, g)$ by putting

$$i_0(f, g) = \dim_{\mathbf{K}} \mathbf{K}[[x, y]]/(f, g),$$

where (f, g) is the ideal of $\mathbf{K}[[x, y]]$ generated by f and g. If f, g are non-zero power series without constant term then $i_0(f, g) < +\infty$ if and only if $\{f = 0\}$ and $\{g = 0\}$ have no common branch. The following properties are basic

1. if Φ is a formal isomorphism then $i_0(f, g) = i_0(f \circ \Phi, g \circ \Phi)$.
2. $i_0(f, gh) = i_0(f, g) + i_0(f, h)$.

Let t be a variable. A *parametrization* is a pair $(\phi(t), \psi(t)) \in \mathbf{K}[[t]]^2$ such that $\phi(t) \neq 0$ or $\psi(t) \neq 0$ in $\mathbf{K}[[t]]$ and $\phi(0) = \psi(0) = 0$. We say that the parametrization $(\phi(t), \psi(t))$ is *good* if the field of fractions of the ring $\mathbf{K}[[\phi(t), \psi(t)]]$ is equal to the field $\mathbf{K}((t))$.

Theorem 2.4 (Normalization Theorem) *Let* $f = f(x, y) \in \mathbf{K}[[x, y]]$ *be an irreducible power series. Then there is a good parametrization* $(\phi(t), \psi(t))$ *such that* $f(\phi(t), \psi(t)) = 0$. *If* $(\alpha(s), \beta(s)) \in \mathbf{K}[[s]]^2$ *is a parametrization such that* $f(\alpha(s), \beta(s)) = 0$ *then there is a power series* $\sigma(s) \in \mathbf{K}[[s]]$, $\sigma(0) = 0$ *such that* $\alpha(s) = \phi(\sigma(s))$ *and* $\beta(s) = \psi(\sigma(s))$.

Hamburger-Noether expansions provide an explicit way to give a parametrization for a branch, valid in any characteristic. Let us recall also

Theorem 2.5 *Under the above assumptions and notations, for any power series* $g = g(x, y) \in \mathbf{K}[[x, y]]$ *we have* $i_0(f, g) = \mathrm{ord}\; g(\phi(t), \psi(t))$.

Taking $g = x$ (respect. $g = y$) we get from the above formula that $\mathrm{ord}\; f(0, y) = i_0(f, x) = \mathrm{ord}\; \phi(t)$ and $\mathrm{ord}\; f(x, 0) = i_0(f, y) = \mathrm{ord}\; \psi(t)$.

Using Theorem 2.5 we check the following two properties of intersection numbers:

3. If f is irreducible, then $i_0(f, g + g') \geq \inf\{i_0(f, g), i_0(f, g')\}$ with equality if $i_0(f, g) \neq i_0(f, g')$.
4. If f is irreducible and $i_0(f, g) = i_0(f, h) < +\infty$ then there exists a constant $c \in \mathbf{K}$ such that $i_0(f, g - ch) > i_0(f, g)$.

In what follows we need

Lemma 2.6 *Let* $f(x, y) \in \mathbf{K}[[x, y]]$ *be an irreducible power series such that* $f(0, y) \neq 0$ *and let* $(\alpha(s), \beta(s))$, $\alpha(s) \neq 0$ *in* $\mathbf{K}[[s]]$, *be a parametrization such that* $f(\alpha(s), \beta(s)) = 0$. *Then, for every power series* $g(x, y) \in \mathbf{K}[[x, y]]$ *we have*

$$\mathrm{ord}\; g(\alpha(s), \beta(s)) = \frac{i_0(f, g)}{i_0(f, x)} \mathrm{ord}\; \alpha(s).$$

Proof Let $(\phi(t), \psi(t))$ be a good parametrization of the branch $\{f(x, y) = 0\}$. Then $\alpha(s) = \phi(\sigma(s))$, $\beta(s) = \psi(\sigma(s))$ for a power series $\sigma(s) \in \mathbf{K}[[s]]$, $\sigma(0) = 0$. We get $\mathrm{ord}\; \alpha(s) = \mathrm{ord}\; \phi(t) \mathrm{ord}\; \sigma(s) = \mathrm{ord}\; f(0, y) \mathrm{ord}\; \sigma(s) = i_0(f, x) \mathrm{ord}\; \sigma(s)$ and consequently

$$\mathrm{ord}\; \sigma(s) = \frac{\mathrm{ord} \alpha(s)}{i_0(f, x)}.$$

On the other hand $\mathrm{ord}\; g(\alpha(s), \beta(s)) = \mathrm{ord}\; g(\phi(t), \psi(t)).\mathrm{ord}\; \sigma(s)$ and by Theorem 2.5 we get

$$\mathrm{ord}\; g(\alpha(s), \beta(s)) = i_0(f, g) \mathrm{ord}\; \sigma(s).$$

Now the formula for $\mathrm{ord}\; g(\alpha(s), \beta(s))$ follows. □

For any irreducible power series $f \in \mathbf{K}[[x, y]]$ we put

$$\Gamma(f) = \{i_0(f, g) \ : \ g \text{ runs over all power series such that } g \not\equiv 0 \ (\mathrm{mod}\, f)\}.$$

Clearly $\Gamma(f)$ is a semigroup. We call $\Gamma(f)$ the *semigroup associated with the branch* $\{f = 0\}$.

Two branches $\{f = 0\}$ and $\{g = 0\}$ are *equisingular* if and only if $\Gamma(f) = \Gamma(g)$. Two formally equivalent branches are equisingular. The branch $\{f = 0\}$ is *non-singular* (that is of multiplicity 1) if and only if $\Gamma(f) = \mathbf{N}$. We have $\min(\Gamma(f)\backslash\{0\}) = \mathrm{ord}\, f$.

Different (but equivalent) definitions of equisingularity were given by Zariski in [47].

Note that the mapping $g \mapsto i_0(f, g)$ induces a valuation v_f of the ring $\mathbf{K}[[x, y]]/(f)$. The semigroup $\Gamma(f)$ can be described as the semigroup of values of v_f.

2.3 The strong triangle inequality

The end of this section is devoted to establish the well-known *Strong Triangle Inequality* given in [35] (see Corollary 3.3) in a slightly more general form and for any characteristic.

Let A be a non-empty set. A function $d : A \times A \longrightarrow \mathbf{R} \cup \{+\infty\}$ satisfying for arbitrary $a, b, c \in A$, the conditions:

(i) $d(a, a) = +\infty$,
(ii) $d(a, b) = d(b, a)$,
(iii) $d(a, b) \geq \inf\{d(a, c), d(b, c)\}$,

will be called a *logarithmic distance* (for short log-distance). We call the third property the *Strong Triangle Inequality* (the STI). It is equivalent to the following

(iii') at least two of the numbers $d(a, b), d(a, c), d(b, c)$ are equal and the third one is not smaller than the other two.

Lemma 2.7 *Let d be a log-distance in the set A. For any $a_1, \ldots, a_m, b_1, \ldots, b_n, c \in A$ at least one of the following conditions holds:*
(I) *there exists $j \in \{1, \ldots, n\}$ such that for any $i \in \{1, \ldots, m\}$, $d(a_i, c) \leq d(a_i, b_j)$,*
(II) *there exists $i \in \{1, \ldots, m\}$ such that for any $j \in \{1, \ldots, n\}$, $d(b_j, c) \leq d(a_i, b_j)$.*

Proof Let us suppose that neither (I) nor (II) holds. Then, for any $j \in \{1, \ldots, n\}$ there exists an index $p(j) \in \{1, \ldots, m\}$ such that $d(a_{p(j)}, c) > d(a_{p(j)}, b_j)$ and, for any $i \in \{1, \ldots, m\}$, there exists $s(i) \in \{1, \ldots, n\}$ such that $d(b_{s(i)}, c) > d(a_i, b_{s(i)})$. Applying the STI to $a_{p(j)}, b_j, c$ and to $a_i, b_{s(i)}, c$ we get

$$d(a_{p(j)}, b_j) = d(b_j, c) < d(a_{p(j)}, c), \tag{1}$$

and

$$d(a_i, b_{s(i)}) = d(a_i, c) < d(b_{s(i)}, c). \tag{2}$$

We may assume without loss of generality that

$$d(a_{p(1)}, b_1) = \sup_{j=1}^{n}\{d(a_{p(j)}, b_j)\}. \tag{3}$$

Using successively (1), (2) and again (1), we get

$$d(a_{p(1)}, b_1) < d(a_{p(1)}, c) = d(a_{p(1)}, b_{s(p(1))}) < d(b_{s(p(1))}, c) = d(a_{p(j_1)}, b_{j_1})$$

with $j_1 = s(p(1))$. Thus we have $d(a_{p(1)}, b_1) < d(a_{p(j_1)}, b_{j_1})$, which contradicts assumption (3). $\square$

An important log-distance on the set of branches can be defined by means of the intersection multiplicity. Let $\{l = 0\}$ be a smooth branch. For any branches $\{f = 0\}$ and $\{g = 0\}$ different from the branch $\{l = 0\}$ we put

$$d_l(f, g) = \frac{i_0(f, g)}{i_0(f, l)i_0(g, l)}.$$

Our aim is to prove

Theorem 2.8 *The function d_l is a log-distance in the set of all branches different from $\{l = 0\}$.*

Proof (cf. [15]) We may assume $l = x$. Since $d_x(f, f) = +\infty$ and $d_x(f, g) = d_x(g, f)$ it suffices to check the STI. Let $\{f = 0\}$, $\{g = 0\}$ and $\{h = 0\}$ be three branches different from $\{x = 0\}$. Let $m = i_0(f, x) = \operatorname{ord} f(0, y)$, $n = i_0(g, x) = \operatorname{ord} g(0, y)$, $p = i_0(h, x) = \operatorname{ord} h(0, y)$. Using the Weierstrass preparation theorem we may assume that f, g, h are distinguished polynomials of degree m, n, p respectively. Using the Normalization Theorem we check (see [41], Theorem 21.18) that there exist power series $\alpha(s)$, $\alpha_i(s)$, $\beta_j(s)$ and $\gamma_k(s)$ such that $f(\alpha(s), y) = \prod_{i=1}^{m}(y - \alpha_i(s))$, $g(\alpha(s), y) = \prod_{j=1}^{n}(y - \beta_j(s))$ and $h(\alpha(s), y) = \prod_{k=1}^{p}(y - \gamma_k(s))$.

The function $d : \mathbf{K}[[s]] \times \mathbf{K}[[s]] \longrightarrow \mathbf{R} \cup \{+\infty\}$ given by $d(\alpha(s), \beta(s)) = \operatorname{ord}(\alpha(s) - \beta(s))$ is a log-distance in $\mathbf{K}[[s]]$. Fix $k \in \{1, \dots, p\}$ and use Lemma 2.7 to $\alpha_1(s), \dots, \alpha_m(s), \beta_1(s), \dots, \beta_n(s)$ and $\gamma(s) = \gamma_k(s)$. Then

(I) there exists $j \in \{1, \dots, n\}$ such that $\operatorname{ord}(\alpha_i(s) - \gamma(s)) \le \operatorname{ord}(\alpha_i(s) - \beta_j(s))$ for all $i \in \{1, \dots, m\}$, or

(II) there exists $i \in \{1, \dots, m\}$ such that $\operatorname{ord}(\beta_j(s) - \gamma(s)) \le \operatorname{ord}(\alpha_i(s) - \beta_j(s))$, for all $j \in \{1, \dots, n\}$.

If (I) holds then $\sum_{i=1}^{m} \operatorname{ord}(\alpha_i(s) - \gamma(s)) \le \sum_{i=1}^{m} \operatorname{ord}(\alpha_i(s) - \beta_j(s))$ that is $\operatorname{ord} f(\alpha(s), \gamma(s)) \le \operatorname{ord} f(\alpha(s), \beta_j(s))$. By Lemma 2.6 we get $\frac{i_0(h,f)}{i_0(x,h)} \le \frac{i_0(g,f)}{i_0(x,g)}$ which implies $d_l(f, h) \le d_l(f, g)$.

If (II) holds then $\sum_{j=1}^{n} \operatorname{ord}(\beta_j(s) - \gamma(s)) \le \sum_{j=1}^{n} \operatorname{ord}(\alpha_i(s) - \beta_j(s))$ that is $\operatorname{ord} g(\alpha(s), \gamma(s)) \le \operatorname{ord} g(\alpha(s), \alpha_i(s))$ and again by Lemma 2.6 we get $\frac{i_0(h,g)}{i_0(x,h)} \le \frac{i_0(f,g)}{i_0(x,f)}$ which implies $d_l(g, h) \le d_l(f, g)$.

Consequently $d_l(f, g) \ge \inf\{d_l(f, h), d_l(g, h)\}$. $\square$

Corollary 2.9 *The function* $d(f,g) = \frac{i_0(f,g)}{\text{ord } f \ \text{ord } g}$ *is a log-distance in the set of all branches.*

Delgado de la Mata in [20] gave an interesting application of Theorem 2.8 to the factorization of *polar curves*.

3 The semigroup of a plane algebroid branch

The aim of this section is to study the structure of the semigroup associated with a plane branch. We follow the method developped by Seidenberg in [40].

Let $f = f(x,y) \in \mathbf{K}[[x,y]]$ be an irreducible power series and let $\Gamma(f)$ be the semigroup associated with the branch $\{f=0\}$. Suppose that $\{f=0\} \neq \{x=0\}$ and put $n = i_0(f,x)$. Let $(\overline{b_0},\dots,\overline{b_h})$, $\overline{b_0} = n$ be the n-minimal system of generators of $\Gamma(f)$.

Lemma 3.1 $\Gamma(f)$ *is a numerical semigroup i.e.* $\gcd(\Gamma(f)) = 1$.

Proof Let $(\phi(t),\psi(t))$ be a good parametrization of the branch $f(x,y) = 0$. Then we have $\mathbf{K}((t)) = \mathbf{K}((\phi(t),\psi(t)))$ and we can write $t = \frac{p(\phi(t),\psi(t))}{q(\phi(t),\psi(t))}$ for some $p(x,y), q(x,y) \in \mathbf{K}[[x,y]]$, $q \not\equiv 0 \pmod f$. Taking orders gives $1 = i_0(f,p) - i_0(f,q)$. Put $a := i_0(f,p)$ and $b := i_0(f,q)$. Then $a, b \in \Gamma(f)$ and $\gcd(a,b) = 1$, which proves the lemma. □

We put $e_0 = n$, $e_k = \gcd(e_{k-1}, \overline{b_k})$ for $k \in \{1,\dots,h\}$ and $n_k = \frac{e_{k-1}}{e_k}$ for $k \in \{1,\dots,h\}$. By Lemma 3.1 we have $e_h = 1$. In what follows we write $v_f(g)$ instead of $i_0(f,g)$.

Theorem 3.2 (Semigroup Theorem) *Let* $\{f=0\}$ *be a branch such that* $\{f=0\} \neq \{x=0\}$. *Set* $n = v_f(x)$ *and let* $\overline{b_0},\dots,\overline{b_h}$ *be the* n*-minimal system of generators of the semigroup* $\Gamma(f)$. *There exists a sequence of monic polynomials* $f_0, f_1, \dots, f_{h-1} \in \mathbf{K}[[x]][y]$ *such that for* $k \in \{1,\dots,h\}$*:*

(a_k *)* $\deg_y(f_{k-1}) = \frac{n}{e_{k-1}}$, *for* $k \in \{1,\dots,h\}$,

(b_k *)* $v_f(f_{k-1}) = \overline{b_k}$ *for* $k \in \{1,\dots,h\}$,

(c_k *)* *if* $k > 1$ *then* $n_{k-1}\overline{b_{k-1}} < \overline{b_k}$.

Moreover $n_k > 1$ *for all* $k \in \{1,\dots,h\}$.

Before giving the proof of the Semigroup Theorem let us note some remarks and corollaries.

The sequence $\overline{b_0},\dots,\overline{b_h}$ is a Seidenberg n-characteristic sequence and will be called the *Seidenberg* n*-characteristic* of the branch $\{f=0\}$ (with respect to the regular branch $\{x=0\}$). We will write $\overline{\text{char}_x} f = (\overline{b_0},\dots,\overline{b_h})$. Therefore $\overline{\text{char}_x} f$ is determined by $n = v_f(x)$ and the semigroup $\Gamma(f)$. Let f_h be the distinguished polynomial associated with f and let $\overline{b_{h+1}} = +\infty$. Then $\deg_y f_h = \frac{n}{e_h} = n$ and $v_f(f_h) = \overline{b_{h+1}} = +\infty$. The polynomials $f_0, f_1, \dots, f_h \in \mathbf{K}[[x]][y]$ will be called *key polynomials* of f. They are not uniquely determined by f.

Corollary 3.3 *Suppose that two branches* $\{f = 0\}$ *and* $\{g = 0\}$ *intersect the axis* $\{x = 0\}$ *with the same multiplicity* $n < +\infty$. *Then* $\overline{\mathrm{char}}_x f = \overline{\mathrm{char}}_x g$ *if and only if* $\{f = 0\}$ *and* $\{g = 0\}$ *are equisingular.*

Let $\overline{\beta_0}, \ldots, \overline{\beta_g}$ be the minimal system of generators of the semigroup $\Gamma(f)$ ($\overline{\beta_0} = \min\{\Gamma(f)\backslash\{0\}\} = \mathrm{ord}\, f$). We put $\overline{\mathrm{char}}\, f = (\overline{\beta_0}, \ldots, \overline{\beta_g})$. Note that $\overline{\mathrm{char}}\, f = \overline{\mathrm{char}}_x f$ if and only if $v_f(x) = \mathrm{ord}\, f$.

The two corollaries presented below follow from the Semigroup Theorem and Proposition 2.3.

Corollary 3.4 (Inversion Formulae) *Let* $\overline{\mathrm{char}}_x f = (\overline{b_0}, \overline{b_1}, \ldots, \overline{b_h})$. *Then* $\overline{\mathrm{char}}\, f = \overline{\mathrm{char}}_x f$ *if and only if* $\overline{b_0} < \overline{b_1}$. *If* $\overline{b_1} < \overline{b_0}$ *and* $\overline{b_0} \not\equiv 0$ *(mod* $\overline{b_1}$*) then* $\overline{\mathrm{char}}\, f = (\overline{b_1}, \overline{b_0}, \ldots, \overline{b_h})$. *If* $\overline{b_0} \equiv 0$ *(mod* $\overline{b_1}$*) then* $\overline{\mathrm{char}}\, f = (\overline{b_1}, \overline{b_2}, \ldots, \overline{b_h})$.

Let $\overline{\mathcal{O}}$ be the normalization of the ring $\mathcal{O} = \mathbf{K}[[x, y]]/(f)$ and let $\mathcal{C}$ be the conductor ideal of $\overline{\mathcal{O}}$ in $\mathcal{O}$. Put $c(f) = \dim_{\mathbf{K}} \overline{\mathcal{O}}/\mathcal{C}$. Then $c(f)$ is the smallest element of $\Gamma(f)$ such that $c(f) + N \in \Gamma(f)$ for any integer $N \geq 0$ (see [11], p. 136).

Corollary 3.5 (Conductor Formula) *If* $\overline{\mathrm{char}}_x f = (\overline{b_0}, \overline{b_1}, \ldots, \overline{b_h})$ *then* $c(f) = \sum_{k=1}^{h}(n_k - 1)\overline{b_k} - \overline{b_0} + 1$.

To prove the Semigroup Theorem let $\{f = 0\}$ be a branch such that $n = i_0(f, x) < +\infty$ and let $\overline{b_0}, \ldots, \overline{b_h}$ be the n-minimal system of generators of the semigroup $\Gamma(f)$. Observe that by the Weierstrass Division Theorem:

$$\Gamma(f) = \{v_f(g) \ : \ g \in \mathbf{K}[[x]][y]\backslash\{0\} \ : \ \deg_y g < n\}.$$

Proposition 3.6 *There exists a monic polynomial* $f_0 \in \mathbf{K}[[x]][y]$ *such that*

(a₁) $\deg_y(f_0) = \frac{n}{e_0} = 1$,

(b₁) $v_f(f_0) = \overline{b_1}$.

To prove Proposition 3.6 we check the following three properties:

Lemma 3.7 (Property I_0) *If* ψ *is a non-zero polynomial with* $\deg_y \psi < 1$ *then* $v_f(\psi) \in \mathbf{N}\overline{b_0}$.

Proof Obviously $\psi \in \mathbf{K}[[x]]$. Thus $v_f(\psi) = (\mathrm{ord}\, \psi)v_f(x) \in \mathbf{N}\overline{b_0}$. □

Lemma 3.8 (Property II_0) *If* $\deg_y \psi < 1$ *then* $v_f(y + \psi) \leq \overline{b_1}$.

Proof Let $g \in \mathbf{K}[[x]][y]$ be such that $v_f(g) = \overline{b_1}$. By the Euclidean division we get $g = Q \cdot (y + \psi) + \psi_1$ with $\psi_1 \in \mathbf{K}[[x]]$. Clearly $v_f(g) \neq v_f(\psi_1)$ and we get $\overline{b_1} \geq \inf\{v_f(g), v_f(\psi_1)\} = v_f(g - \psi_1) = v_f(Q \cdot (y + \psi)) \geq v_f(y + \psi)$. □

Lemma 3.9 (Property III_0) *If* $\psi \in \mathbf{K}[[x]]$ *and* $v_f(y + \psi) \in \mathbf{N}\overline{b_0}$ *then there exists a power series* $\overline{\psi} \in \mathbf{K}[[x]]$ *such that* $v_f(y + \overline{\psi}) > v_f(y + \psi)$.

Proof There exists an integer $a \geq 0$ such that $v_f(y + \psi) = a\overline{b_0} = v_f(x^a)$. Therefore there is an element $c \in \mathbf{K}$ such that $v_f(y + \psi - cx^a) > v_f(y + \psi)$. We put $\overline{\psi} = \psi - cx^a$. □

Proof of Proposition 2.6 From Properties II_0 and III_0 it follows that there exists a monic polynomial f_0 of degree 1 such that $v_f(f_0) \notin \mathbf{N}\overline{b_0}$. By definition of $\overline{b_1}$ we get $v_f(f_0) \geq \overline{b_1}$. The equality follows from Property II_0. □

Proposition 3.10 *Suppose that there exist monic polynomials* $f_0, f_1, \ldots, f_{k-1}$ *in* $\mathbf{K}[[x]][y]$ *such that*

(a_i) $\deg_y(f_{i-1}) = \frac{n}{e_{i-1}}$, *for* $i \in \{1, \ldots, k\}$,

(b_i) $v_f(f_{i-1}) = \overline{b_i}$ *for* $i \in \{1, \ldots, k\}$,

(c_i) $n_{i-1}\overline{b_{i-1}} < \overline{b_i}$ *for* $i \in \{2, \ldots, k\}$.

Then there exists a monic polynomial $f_k \in \mathbf{K}[[x]][y]$ *such that*

(a_{k+1}) $\deg_y(f_k) = \frac{n}{e_k}$,

(b_{k+1}) $v_f(f_k) = \overline{b_{k+1}}$,

(c_{k+1}) $n_k\overline{b_k} < \overline{b_{k+1}}$.

To prove Proposition 3.10 we check the following three properties:

Lemma 3.11 (Property I_k) *If* ψ *is a non-zero polynomial with* $\deg_y \psi < \frac{n}{e_k}$ *then* $v_f(\psi) \in \mathbf{N}\overline{b_0} + \cdots + \mathbf{N}\overline{b_k}$.

Proof Let $l \leq k$. We will prove that for $\deg_y \psi < \frac{n}{e_l}$ we have $v_f(\psi) \in \mathbf{N}\overline{b_0} + \cdots + \mathbf{N}\overline{b_l}$. We proceed by induction on l. The case $l = 0$ is already proved (see Property I_0). Let $l > 0$ and suppose the property holds for polynomials of degree less than $\frac{n}{e_{l-1}}$. Fix $\psi \in \mathbf{K}[[x]][y]$ with $\deg_y(\psi) < \frac{n}{e_l}$ and consider the f_{l-1}-adic expansion of ψ:

$$\psi = \psi_0 f_{l-1}^s + \psi_1 f_{l-1}^{s-1} + \cdots + \psi_s, \tag{4}$$

where $\psi_0 \neq 0$, $\deg_y(\psi_i) < \deg_y(f_{l-1}) = \frac{n}{e_{l-1}}$.

Note that $s \leq \frac{\deg_y(\psi)}{\deg_y(f_{l-1})} < n_l$. Let I be the set of all $i \in \{0, \ldots, s\}$ such that $\psi_i \neq 0$. Therefore, by the induction hypothesis we get $v_f(\psi_i) \in \mathbf{N}\overline{b_0} + \cdots + \mathbf{N}\overline{b_{l-1}}$, and

$$v_f(\psi_i) \equiv 0 \mod e_{l-1} \quad \text{for } i \in I. \tag{5}$$

Moreover

$$v_f(\psi_i f_{l-1}^{s-i}) \neq v_f(\psi_j f_{l-1}^{s-j}) \quad \text{for } i \neq j \in I. \tag{6}$$

Indeed, suppose that (6) is not true, so there exist $i, j \in I$ such that $i < j$ and $v_f(\psi_i f_{l-1}^{s-i}) = v_f(\psi_j f_{l-1}^{s-j})$. Therefore $v_f(\psi_i) + (s-i)v_f(f_{l-1}) = v_f(\psi_j) + (s-j)v_f(f_{l-1})$ and $(j-i)\overline{b_l} = v_f(\psi_j) - v_f(\psi_i) \equiv 0 \mod e_{l-1}$ by (5). The last relation implies $(j-i)\frac{\overline{b_l}}{e_l} \equiv 0 \mod n_l$ and consequently $j - i \equiv 0 \mod n_l$ because $\frac{\overline{b_l}}{e_l}$ and n_l are co-prime. We get a contradiction because $0 < j - i \leq s < n_l$. Now by (4) and (6) we get

$$\begin{aligned} v_f(\psi) &= \min_{i=0}^{s} v_f(\psi_i f_{l-1}^{s-i}) = v_f(\psi_j f_{l-1}^{s-j}) \\ &= v_f(\psi_j) + (s-j)\overline{b_l} \in \mathbf{N}\overline{b_0} + \cdots + \mathbf{N}\overline{b_l}, \end{aligned}$$

for some $j \in I$. □

Lemma 3.12 (Property II$_k$) *If* $\deg_y \psi < \frac{n}{e_k}$ *then* $v_f(y^{\frac{n}{e_k}} + \psi) \leq \overline{b_{k+1}}$.

Proof Let $g \in \mathbf{K}[[x]][y]$ be such that $v_f(g) = \overline{b_{k+1}}$. By the Euclidean division we get $g = Q \cdot (y^{\frac{n}{e_k}} + \psi) + \psi_1$ with $\psi_1 \in \mathbf{K}[[x]][y]$ and $\deg_y \psi_1 < \frac{n}{e_k}$. We may assume $\psi_1 \neq 0$. Therefore $v_f(\psi_1) \in \mathbf{N}\overline{b_0} + \cdots + \mathbf{N}\overline{b_k}$ by Property I$_k$ and $v_f(g) = \overline{b_{k+1}} \neq v_f(\psi_1)$. Now we get $\overline{b_{k+1}} \geq \inf\{v_f(g), v_f(\psi_1)\} = v_f(g - \psi_1) = v_f(Q \cdot (y^{\frac{n}{e_k}} + \psi)) \geq v_f(y^{\frac{n}{e_k}} + \psi)$. □

Lemma 3.13 (Property III$_k$) *If* $\psi \in \mathbf{K}[[x]][y]$ *with* $\deg_y \psi < \frac{n}{e_k}$ *and* $v_f(y^{\frac{n}{e_k}} + \psi) \in \mathbf{N}\overline{b_0} + \cdots + \mathbf{N}\overline{b_k}$ *then there is a polynomial* $\overline{\psi} \in \mathbf{K}[[x]][y]$, $\deg_y \overline{\psi} < \frac{n}{e_k}$ *such that* $v_f(y^{\frac{n}{e_k}} + \overline{\psi}) > v_f(y^{\frac{n}{e_k}} + \psi)$.

Proof By Proposition 2.1 any element of the semigroup $\mathbf{N}\overline{b_0} + \cdots + \mathbf{N}\overline{b_k}$ has the form $a_0\overline{b_0} + a_1\overline{b_1} + \cdots + a_k\overline{b_k}$ with $a_0 \geq 0$ and $0 \leq a_i < n_i$ for $i \in \{1, \ldots, k\}$. Therefore we can write $v_f(y^{\frac{n}{e_k}} + \psi) = v_f(x^{a_0} f_0^{a_1} \cdots f_{k-1}^{a_k})$ and there is an element $c \in \mathbf{K}$ such that $v_f(y^{\frac{n}{e_k}} + \psi - c x^{a_0} f_0^{a_1} \cdots f_{k-1}^{a_k}) > v_f(y^{\frac{n}{e_k}} + \psi)$. Let $\overline{\psi} = \psi - c x^{a_0} f_0^{a_1} \cdots f_{k-1}^{a_k}$. Then we have $v_f(y^{\frac{n}{e_k}} + \overline{\psi}) > v_f(y^{\frac{n}{e_k}} + \psi)$. Since $\deg_y(x^{a_0} f_0^{a_1} \cdots f_{k-1}^{a_k}) = a_1 + a_2 \frac{n}{e_1} + \cdots + a_k \frac{n}{e_{k-1}} \leq (n_1 - 1) + (n_2 - 1)\frac{n}{e_1} + \cdots + (n_k - 1)\frac{n}{e_{k-1}} = \frac{n n_k}{e_{k-1}} - 1 < \frac{n}{e_k}$, $\deg_y \overline{\psi} < \frac{n}{e_k}$. □

Proof of Proposition 3.10 From Properties II$_k$ and III$_k$ it follows that there exists a monic polynomial f_k of degree $\frac{n}{e_k}$ such that $v_f(f_k) \notin \mathbf{N}\overline{b_0} + \cdots + \mathbf{N}\overline{b_k}$. By definition of $\overline{b_{k+1}}$ we get $v_f(f_k) \geq \overline{b_{k+1}}$. The equality follows from Property II$_k$.

To check (c_{k+1}) observe that $v_f(f_{k-1}^{n_k}) = n_k\overline{b_k}$ and $\deg_y f_{k-1}^{n_k} = n_k \frac{n}{e_{k-1}} = \frac{n}{e_k}$. Therefore $n_k\overline{b_k} \leq \overline{b_{k+1}}$ by Property II$_k$ and we get $n_k\overline{b_k} < \overline{b_{k+1}}$ since $\overline{b_{k+1}} \notin \mathbf{N}\overline{b_0} + \cdots + \mathbf{N}\overline{b_k}$. □

Proof of Theorem 3.2 The theorem follows by induction from Proposition 3.6, Proposition 3.10 and from Remark 2.2. □

Notes

Seidenberg gave in [40] the description of the semigroup of a zero-dimensional valuation of the extension $\mathbf{K}(x, y)/\mathbf{K}$ ([40], Theorem 6, p. 398) in terms of generators. The case of the semigroup associated with an algebroid plane branch was studied by Azevedo in [9]. His method based on the Apéry sequences was extended by Angermüller in [7] to the case of arbitrary characteristic. For different characterizations of the numerical semigroups we refer the reader to [26], Chapter 6.

If $n = v_f(x) \not\equiv 0 \pmod{\operatorname{char} \mathbf{K}}$ the Puiseux series are available. Zariski in [48] (see also [25,36]) constructed the sequence $\overline{\beta_0}, \ldots, \overline{\beta_g}$ and the corresponding sequence of key polynomials by using Puiseux series expansion determined by the equation $f(x, y) = 0$. This method turned out efficient when applied to the semigroups of integers associated with meromorphic curves (see [2–4]). A proof of the Semigroup Theorem based on the Hamburger-Noether expansion was given by Russel in [38] and Campillo in [11,12]. To describe the semigroup of a plane branch one can use the *characteristic pairs* (see [7,29,32]) instead of the generators.

4 Key polynomials and approximate roots

The key polynomials under the name of semi-roots were studied by Abhyankar [1] and Popescu-Pampu [36]. Here we propose the treatment without any restriction on the field characteristic.

Let $f = f(x, y) \in \mathbf{K}[[x, y]]$ be an irreducible power series such that $i_0(f, x) = \operatorname{ord} f(0, y) = n < +\infty$ and let $\overline{\operatorname{char}}_x f = (\overline{b_0}, \ldots, \overline{b_h})$, $\overline{b_0} = n$. Let $k \in \{0, \ldots, h\}$. Recall that a monic polynomial $g \in \mathbf{K}[[x]][y]$ is a k-th *key polynomial* of f if $\deg_y g = \frac{n}{e_k}$ and $v_f(g) = \overline{b_{k+1}}$. By the Semigroup Theorem, for any $k \in \{0, \ldots, h\}$ there exists a k-th key polynomial of f.

Lemma 4.1 *Let $f = f(x, y) \in \mathbf{K}[[x, y]]$ be an irreducible power series such that $n = i_0(f, x) < +\infty$ and let $(\overline{b_0}, \ldots, \overline{b_h})$ be an n-characteristic sequence. Suppose that there exist monic polynomials $f_0, \ldots, f_{h-1} \in \mathbf{K}[[x]][y]$ such that $\deg_y f_k = \frac{n}{e_k}$ and $i_0(f, f_k) = \overline{b_{k+1}}$ for $k \in \{0, \ldots, h-1\}$. Then $\overline{\operatorname{char}}_x f = (\overline{b_0}, \ldots, \overline{b_h})$ and $f_0, \ldots, f_{h-1}$ are key polynomials of f.*

Proof Recall that $\Gamma(f) = \{v_f(g) \ : \ g \in \mathbf{K}[[x]][y]\backslash\{0\} \ : \ \deg_y g < n\}$. By Lemma 3.11 we get $\Gamma(f) = \mathbf{N}\overline{b_0} + \cdots + \mathbf{N}\overline{b_h}$. According to the first statement of Proposition 2.3 the sequence $\overline{b_0}, \ldots, \overline{b_h}$ is the n-minimal system of generators of the semigroup $\Gamma(f)$ and the lemma follows. □

Proposition 4.2 *Let g be a k-th key polynomial of f. Then g is a distinguished polynomial, irreducible in $\mathbf{K}[[x, y]]$ with characteristic $\overline{\operatorname{char}}_x g = \left(\frac{\overline{b_0}}{e_k}, \ldots, \frac{\overline{b_k}}{e_k}\right)$. Moreover the polynomials $f_0, f_1, \ldots, f_{k-1}$ are key polynomials of g.*

Proof The polynomial g is irreducible since the value $\overline{b_{k+1}} = v_f(g)$ is irreducible in the semigroup $\Gamma(f)$. Moreover, $i_0(x, g) = \frac{n}{e_k}$ that is g is a distinguished polynomial. If we had $i_0(x, g) < \frac{n}{e_k}$ then g would be associated with a distinguished polynomial of degree less than $\frac{n}{e_k}$, which contradicts Property I_k. Let us calculate $i_0(f_i, g)$ for $i < k$. Consider f_i, g, f and the log-distances $d_x(f_i, g) = \frac{e_i e_k i_0(f_i, g)}{n^2}$, $d_x(f_i, f) = \frac{e_i \overline{b_{i+1}}}{n^2}$ and $d_x(g, f) = \frac{e_k \overline{b_{k+1}}}{n^2}$. The sequence $\left(e_{i-1}\overline{b_i}\right)$ is strictly increasing, therefore $d_x(f_i, f) < d_x(g, f)$ and by the STI we get $d_x(f_i, g) = d_x(f_i, f)$ which implies $i_0(f_i, g) = \frac{\overline{b_{i+1}}}{e_k}$.

On the other hand $\deg_y f_i = \frac{n}{e_i} = \frac{n}{e_k} : \frac{e_i}{e_k}$ and $\frac{e_i}{e_k} = \gcd\left(\frac{\overline{b_0}}{e_k}, \ldots, \frac{\overline{b_i}}{e_k}\right)$. The proposition follows from Lemma 4.1. □

Proposition 4.3 *Let $h \in \mathbf{K}[[x]][y]$ be a $(k-1)$-th key polynomial of f and let $g \in \mathbf{K}[[x]][y]$ be a monic polynomial such that $\deg_y g = \frac{n}{e_k}$ and $v_f(g) > n_k\overline{b_k}$. Let $g = h^{n_k} + a_1 h^{n_k - 1} + \cdots + a_{n_k}$, $\deg_y a_i < \deg_y h = \frac{n}{e_{k-1}}$ be the h-adic expansion of g. Then $v_f(a_i) > i\overline{b_k}$ if $1 \leq i < n_k$ and $v_f(a_{n_k}) = n_k\overline{b_k}$.*

Proof Consider the h-adic expansion of g

$$g = h^{n_k} + a_1 h^{n_k - 1} + \cdots + a_{n_k}, \tag{7}$$

where $\deg_y a_i < \deg_y h = n/e_{k-1}$.

Let I be the set of all $i \in \{1, \ldots, n_k\}$ such that $a_i \neq 0$. Since $v_f(g) > n_k \overline{b_k} = v_f(h^{n_k})$, $I \neq \emptyset$. There is $v_f(a_i) < +\infty$ for $i \in I$ and by Property I_k we get $v_f(a_i) \in \mathbf{N}\overline{b_0} + \cdots + \mathbf{N}\overline{b_{k-1}}$, hence $v_f(a_i) \equiv 0 \bmod e_{k-1}$ for every $i \in I$. We have

$$v_f(a_i h^{n_k - i}) \neq v_f(a_j h^{n_k - j}) \tag{8}$$

for $i, j \in I$ with $i \neq j$.

Indeed, $v_f(a_i h^{n_k - i}) = v_f(a_j h^{n_k - j})$ with $i < j$ implies, as in the proof of Property I_k, the congruence $(j - i)\overline{b_k}/e_k \equiv 0 \bmod n_k$, which leads to a contradiction for $0 < j - i < n_k$.

From (7) and (8) we have

$$v_f(g - h^{n_k}) = \min_{i=1}^{n_k} v_f(a_i h^{n_k - i}). \tag{9}$$

By assumption $v_f(g) > n_k \overline{b_k} = v_f(h^{n_k})$, so $v_f(g - h^{n_k}) = n_k \overline{b_k}$ and (9) implies $n_k \overline{b_k} \leq v_f(a_i h^{n_k - i}) = v_f(a_i) + (n_k - i)\overline{b_k}$ for $i \in \{1, \ldots, n_k\}$. Therefore we get

$$v_f(a_i) \geq i\overline{b_k} \tag{10}$$

for $i \in \{1, \ldots, n_k\}$.

Moreover,

$$\text{if } v_f(a_i) = i\overline{b_k} \text{ for } i \in \{1, \ldots, n_k\} \text{ then } i = n_k. \tag{11}$$

Indeed, from $v_f(a_i) = i\overline{b_k}$ it follows that $i\overline{b_k} \equiv 0 \bmod e_{k-1}$ and $i\overline{b_k}/e_k \equiv 0 \bmod n_k$, so $i \equiv 0 \bmod n_k$ because $\overline{b_k}/e_k$ and n_k are coprime. Hence we get $i = n_k$. According to (9) there exists $i_0 \in I$ such that $v_f(a_{i_0} h^{n_k - i_0}) = v_f(g - h^{n_k}) = n_k \overline{b_k}$. Thus $v_f(a_{i_0}) = i_0 \overline{b_k}$ and by (11) we get $i_0 = n_k$. □

To prove the Abhyankar–Moh Theorem on approximate roots we use the properties of key polynomials explained above. First let us recall the basic notions of Abhyankar–Moh theory (see [2–4] or [36]).

Let R be an integral domain and let $d > 1$ be a positive integer such that d is a unit in R. Denote $\deg f := \deg_y f$ the degree of the polynomial $f \in R[y]$ in one variable y and assume that d divides $\deg f$. According to Abhyankar and Moh ([3,4], Sect. 1) the *approximate d-th* root of f, denoted by $\sqrt[d]{f}$ is defined to be the unique monic polynomial satisfying $\deg(f - (\sqrt[d]{f})^d) < \deg f - \deg \sqrt[d]{f}$. For the existence and uniqueness of $\sqrt[d]{f}$ see [3,4]. We put by convention $\sqrt[1]{f} = f$. Obviously $\deg \sqrt[d]{f} = \frac{\deg f}{d}$. From the definition it follows that $\sqrt[e]{\sqrt[d]{f}} = \sqrt[ed]{f}$ if ed is a unit which divides $\deg f$ (see [25]).

Given any monic polynomial $g \in R[y]$ of degree $\deg f/d$ we have the g-adic expansion of f, namely

$$f = g^d + a_1 g^{d-1} + \cdots + a_d,$$

where $a_i \in R[y]$, $\deg a_i < \deg g$.

The polynomials a_i are uniquely determined by f and g.

The *Tschirnhausen operator* $\tau_f(g) := g + \frac{1}{d}a_1$ maps g to $\tau_f(g)$ which is again a monic polynomial of degree $\deg f/d$. One checks (see [3,4], Sect. 1 and Sect. 6) that

1. $a_1 = 0$ if and only if $g = \sqrt[d]{f}$,
2. if $f = (\tau_f(g))^d + \overline{a_1}(\tau_f(g))^{d-1} + \cdots + \overline{a_d}$ is the $\tau_f(g)$-expansion of f then $\deg \overline{a_1} < \deg a_1$ or $\overline{a_1} = 0$.

   Using the above properties we get
3. $\sqrt[d]{f} = \tau_f(\tau_f \cdots (\tau_f(g)))$ with τ_f repeated $\deg f/d$ times.

Let $f = f(x, y) \in \mathbf{K}[[x]][y]$ be an irreducible distinguished polynomial of degree $n > 1$ such that $\overline{\text{char}_x f} = (\overline{b_0}, \ldots, \overline{b_h})$, $\overline{b_0} = n$.

Proposition 4.4 *Let* $g = g(x, y) \in \mathbf{K}[[x]][y]$ *be a monic polynomial such that* $\deg_y g = \frac{n}{e_k}$ *and* $v_f(g) > n_k\overline{b_k}$. *Assume that* $n_k \not\equiv 0$ *mod char* $\mathbf{K}$. *Then*

(i) if h *is a* $(k-1)$*-th key polynomial of* f *then* $\tau_g(h)$ *is a* $(k-1)$*-th key polynomial of* f *as well,*

(ii) $v_f(\sqrt[n_k]{g}) = \overline{b_k}$.

Proof Consider the h-adic expansion of g: $g = h^{n_k} + a_1 h^{n_k-1} + \cdots + a_{n_k}$. By Proposition 4.3 we get $v_f(a_1) > \overline{b_k}$ (because $n_k > 1$). Therefore $v_f(\tau_g(h)) = v_f(h + \frac{1}{n_k}a_1) = v_f(h) = \overline{b_k}$. Clearly $\deg_y \tau_g(h) = \deg_y h$ and (i) follows.

To check (ii) use $\deg_y g/n_k = n/e_{k-1}$ times (i) and the formula for the approximate root $\sqrt[n_k]{g}$ in terms of τ_g. □

Now we can prove the Abhyankar–Moh Theorem (see [3,4]).

Theorem 4.5 (Abhyankar-Moh Fundamental Theorem on approximate roots) *Let* $f = f(x, y) \in \mathbf{K}[[x]][y]$ *be an irreducible distinguished polynomial of degree* $n > 1$ *with* $\overline{\text{char}_x f} = (\overline{b_0}, \overline{b_1}, \ldots, \overline{b_h})$ *and* $\overline{b_0} = v_f(x) = n$. *Let* $1 \leq k \leq h+1$. *Suppose that* $e_{k-1} \not\equiv 0$ *mod char* $\mathbf{K}$. *Then:*

1. $v_f(\sqrt[e_{k-1}]{f}) = \overline{b_k}$,

2. $\sqrt[e_{k-1}]{f}$ *is an irreducible distinguished polynomial of degree* n/e_{k-1} *such that* $\overline{\text{char}_x \sqrt[e_{k-1}]{f}} = (\overline{b_0}/e_{k-1}, \overline{b_1}/e_{k-1}, \ldots, \overline{b_{k-1}}/e_{k-1})$.

Proof According to Proposition 4.2 it suffices to check the first part of the theorem. We use descendent induction on k. If $k = h+1$ then $e_{k-1} = e_h = 1$, $\overline{b_k} = \overline{b_{h+1}} = +\infty$ and obviously $v_f(\sqrt[e_h]{f}) = \overline{b_{h+1}}$. Let $k \leq h$. Suppose that $e_k \not\equiv 0$ (mod char $\mathbf{K}$) and $v_f(\sqrt[e_k]{f}) = \overline{b_{k+1}}$. The polynomial $\sqrt[e_k]{f}$ is of degree n/e_k and $v_f(\sqrt[e_k]{f}) > n_k\overline{b_k}$ so we can apply Proposition 4.4 (ii) to $g = \sqrt[e_k]{f}$ to get $v_f(\sqrt[n_k]{g}) = \overline{b_k}$ provided that $n_k \not\equiv 0$ (mod char $\mathbf{K}$).

Assume that $e_{k-1} \not\equiv 0$ (mod char $\mathbf{K}$). Then $e_k, n_k \not\equiv 0$ (mod char $\mathbf{K}$) and we have $\sqrt[n_k]{g} = \sqrt[n_k]{\sqrt[e_k]{f}} = \sqrt[e_{k-1}]{f}$. Consequently, $v_f(\sqrt[e_{k-1}]{f}) = \overline{b_k}$ and we are done. □

Corollary 4.6 *Suppose that $n \not\equiv 0$ (mod char $\mathbf{K}$). Then $\sqrt[e_0]{f}$, $\sqrt[e_1]{f}, \dots, \sqrt[e_h]{f}$ is a sequence of key polynomials of f.*

Notes

S. Maclane introduced key polynomials in the classical work [30]. The maximal contact curves (which are exactly the same concept) were studied by M. Lejeune-Jalabert in [29] (see also [11] and [21] (Appendix E)) in positive characteristic. The key polynomials under the name of semiroots appeared also in Abhyankar's paper [1] (see [Po] for detailed treatment in characteristic 0). They are connected with *curvettes* associated with extremal points in the dual graph of $\{f = 0\}$ (see, for example [22] p. 54, [36] p.13). They also play an important role in studying valuations [42].

S.S. Abhyankar and T.T. Moh developed the theory of approximate roots of polynomials with coefficients in the meromorphic series field $\mathbf{K}((x))$ in the fundamental paper [3,4]. In [5] they applied approximate roots to prove the Embedding Line Theorem. Later on Abhyankar in [2] gave a simplified version of [3,4] and [5]. The approach of Abhyankar and Moh is based on the technique of deformations of power series. H. Pinkham in [34] proposed a method of eliminating the deformations which works in the *algebroid case* $\mathbf{K}[[x]][y]$. P. Russel in [38] used the Hamburger-Noether expansions to reprove the Abhyankar-Moh results (in the algebroid case) with weaker assumptions on the field characteristic. In our presentation of the subject we followed [25] (see also[16,17]). The reader will find in [36] more references on the approximate roots.

5 The Abhyankar–Moh irreducibility criterion

The aim of this section is to give a version of the Abhyankar–Moh irreducibility criterion. Our proof is based on Theorem 5.2 below.

Let $\{f = 0\}$ and $\{g = 0\}$ be two branches different from $\{x = 0\}$. Let $\overline{\mathrm{char}_x} f = (\overline{b_0}, \dots, \overline{b_h})$, $\overline{b_0} = n = i_0(f, x)$ and $\overline{\mathrm{char}_x} g = (\overline{b'_0}, \dots, \overline{b'_{h'}})$, $\overline{b'_0} = n' = i_0(g, x)$. We denote by $f_0, \dots, f_h$ and $g_0, \dots, g_{h'}$ key polynomials of f and g, respectively.

Lemma 5.1 *The equalities $\frac{\overline{b_i}}{n} = \frac{\overline{b'_i}}{n'}$ for all $i \in \{1, \dots, k\}$ imply $\frac{n}{e_i} = \frac{n'}{e'_i}$ and $\frac{\overline{b_i}}{e_i} = \frac{\overline{b'_i}}{e'_i}$ for all $i \in \{1, \dots, k\}$.*

Proof We get $ne'_i = n \gcd(\overline{b'_0}, \dots, \overline{b'_i}) = \gcd(n\overline{b'_0}, \dots, n\overline{b'_i}) = n'e_i$. Thus $\frac{n}{e_i} = \frac{n'}{e'_i}$ and consequently $\frac{\overline{b_i}}{e_i} = \frac{\overline{b'_i}}{e'_i}$ for all $i \in \{1, \dots, k\}$ since $\frac{\overline{b_i}}{n} = \frac{\overline{b'_i}}{n'}$. □

Theorem 5.2 *Let $n = i_0(f, x) > 1$ and suppose that $\frac{i_0(f,g)}{i_0(x,g)} > \frac{e_{k-1}\overline{b_k}}{n}$ for an integer $k \in \{1, \dots, h\}$. Then $k \le h'$ and $\frac{\overline{b_i}}{n} = \frac{\overline{b'_i}}{n'}$ for all $i \in \{1, \dots, k\}$. The first k key polynomials $f_0, \dots, f_{k-1}$ of f are the first k key polynomials of g.*

Proof Let us start with

$$n i_0(g, f_{i-1}) = n'\overline{b_i} \text{ for } i \in \{1, \dots, k\}. \tag{12}$$

Fix $i \in \{1, \ldots, k\}$ and consider the power series f, f_{i-1} and g. We have $d_x(f, f_{i-1}) = \frac{e_{i-1}\overline{b_i}}{n^2}$, $d_x(f, g) = \frac{i_0(f,g)}{nn'} > \frac{e_{k-1}\overline{b_k}}{n^2}$ (by assumption) and $d_x(g, f_{i-1}) = \frac{e_{i-1}i_0(g,f_{i-1})}{nn'}$. Since $d_x(f, f_{i-1}) < d_x(f, g)$ by the STI we get $d_x(g, f_{i-1}) = d_x(f, f_{i-1})$, which implies (12).

Remark that

$$n' \equiv 0 \left(\mathrm{mod} \frac{n}{e_k}\right). \tag{13}$$

Indeed, we may write $e_k = a_0\overline{b_0} + a_1\overline{b_1} + \cdots + a_k\overline{b_k}$ with $a_0, \ldots, a_k \in \mathbf{Z}$ since $e_k = \gcd(\overline{b_0}, \ldots, \overline{b_k})$. Hence we get $e_k n' = (a_0 n')n + a_1(n'\overline{b_1}) + \cdots + a_k(n'\overline{b_k}) \equiv 0$ (mod n) by (12) and consequently $n' \equiv 0$ (mod $\frac{n}{e_k}$). □

Property 5.3 *Let $i > 0$ be an integer. Then $d_x(g, f_{i-1}) = \frac{e_{i-1}\overline{b_i}}{n^2}$ for $i \leq k$, $d_x(g, g_{i-1}) = \frac{e'_{i-1}\overline{b'_i}}{(n')^2}$ for $i \leq h'$, and $d_x(f_{i-1}, g_{i-1}) = \frac{e_{i-1}e'_{i-1}i_0(f_{i-1},g_{i-1})}{nn'}$ for $i \leq \min(k, h')$.*

Proof We have $d_x(g, f_{i-1}) = \frac{e_{i-1}i_0(g,f_{i-1})}{n'n} = \frac{e_{i-1}\overline{b_i}}{n^2}$ by (12). The formulae for $d_x(g, g_{i-1})$ and $d_x(f_{i-1}, g_{i-1})$ follow from the definitions. □

We have

$$h' \geq 1 \quad \text{and} \quad \frac{\overline{b_1}}{n} = \frac{\overline{b'_1}}{n'}. \tag{14}$$

From (13) it follows that $n' > 1$ since $\frac{n}{e_k} > 1$ for $k > 0$. Thus $h' \geq 1$ and we may apply Property 5.3 for $i = 1$. We get $d_x(g, f_0) = \frac{\overline{b_1}}{n} \notin \mathbf{N}$, $d_x(g, g_0) = \frac{\overline{b'_1}}{n'} \notin \mathbf{N}$ and $d_x(f_0, g_0) = i_0(f_0, g_0) \in \mathbf{N}$. By the STI we obtain $\frac{\overline{b_1}}{n} = \frac{\overline{b'_1}}{n'}$.

Property 5.4 *Let $i > 0$ be an integer such that $i < k$, $i \leq h'$ and $\frac{\overline{b_j}}{n} = \frac{\overline{b'_j}}{n'}$ for all $j \leq i$. Then $i < h'$ and $\frac{\overline{b_{i+1}}}{n} = \frac{\overline{b'_{i+1}}}{n'}$.*

Proof From the assumption $\frac{\overline{b_j}}{n} = \frac{\overline{b'_j}}{n'}$ for all $j \leq i$ and from Lemma 5.1 we get $\frac{e_i}{n} = \frac{e'_i}{n'}$. By (13) we may write $n' = l\frac{n}{e_k}$, where $l > 0$ is an integer. Thus $e'_i = n'\frac{e_i}{n} = l\frac{e_i}{e_k} > 1$ since $i < k$. From $e'_i > 1$ we get obviously $i < h'$. Now we may apply Property 5.3 for the index $i + 1$ since $i + 1 \leq k$ and $i + 1 \leq h'$. We get $d_x(g, f_i) = \frac{e_i\overline{b_{i+1}}}{n^2}$, $d_x(g, g_i) = \frac{e'_i\overline{b'_{i+1}}}{(n')^2}$ and $d_x(f_i, g_i) = \left(\frac{e_i}{n}\right)\left(\frac{e'_i}{n'}\right) i_0(f_i, g_i)$. Recall that $\frac{e_i}{n} = \frac{e'_i}{n'}$. Note that $d_x(g, f_i) \neq d_x(f_i, g_i)$. Indeed if we had $d_x(g, f_i) = d_x(f, g_i)$ then we would get $\overline{b_{i+1}} = e_i i_0(f_i, g_i)$ which is impossible since $\overline{b_{i+1}} \not\equiv 0$ (mod e_i). Similarly we check that $d_x(g, g_i) \neq d_x(f_i, g_i)$. Using the STI we get $d_x(g, f_i) = d_x(g, g_i)$, which implies $\frac{\overline{b_{i+1}}}{n} = \frac{\overline{b'_{i+1}}}{n'}$. □

Now we can finish the proof of Theorem 5.2.

From Properties (14) and 5.4 we conclude that $k \leq h'$ and $\frac{\overline{b_i}}{n} = \frac{\overline{b'_i}}{n'}$ for $i \in \{1, \ldots, k\}$, which proves the first part of Theorem 5.2. Let $i \in \{0, 1, \ldots, k-1\}$. By

Property (12) $i_0(g, f_{i-1}) = \frac{n'\overline{b_i}}{n} = \overline{b_i'}$ since $\frac{\overline{b_i}}{n} = \frac{\overline{b_i'}}{n'}$. Moreover $\deg_y(f_{i-1}) = \frac{n}{e_{i-1}} = \frac{n'}{e'_{i-1}}$ so f_{i-1} is a key-polynomial of g. □

Remark 5.5 Under the notations and assumptions of Theorem 5.2, we get

$$\frac{i_0(g, f)}{i_0(x, f)} = \frac{i_0(f, g)}{i_0(x, g)}\frac{i_0(x, g)}{i_0(x, f)} > \frac{e_{k-1}\overline{b_k}}{n}\frac{n'}{n} = \frac{e'_{k-1}}{n'}\frac{\overline{b_k'}}{n'}n' = \frac{e'_{k-1}\overline{b_k'}}{n'}.$$

Let $f \in \mathbf{K}[[x, y]]$ be an irreducible power series such that $\{f = 0\} \neq \{x = 0\}$. Let $n = i_0(f, x) > 1$ and $\overline{\text{char}_x} f = (\overline{b_0}, \ldots, \overline{b_h})$, $\overline{b_0} = n$.

Lemma 5.6 *Let $g = g(x, y) \in \mathbf{K}[[x, y]]$ be an irreducible power series such that $\{g = 0\} \neq \{x = 0\}$ and let k be an integer such that $1 \leq k \leq h$. If $\frac{i_0(f,g)}{i_0(g,x)} > \frac{e_{k-1}\overline{b_k}}{n}$ then $i_0(g, x) \equiv 0 \left(mod\ \frac{n}{e_k}\right)$. If, additionally, $i_0(g, x) = \frac{n}{e_k}$ then $\overline{\text{char}_x} g = \left(\frac{\overline{b_0}}{e_k}, \ldots, \frac{\overline{b_k}}{e_k}\right)$.*

Proof The lemma follows from Theorem 5.2 and Lemma 5.1. □

Theorem 5.7 *Let $g \in \mathbf{K}[[x, y]]$ be a power series such that $i_0(g, x) = \frac{n}{e_k}$ and $i_0(f, g) > n_k\overline{b_k}$ for a $k \in \{1, \ldots, h\}$. Then g is irreducible and $\overline{\text{char}_x} g = \left(\frac{\overline{b_0}}{e_k}, \ldots, \frac{\overline{b_k}}{e_k}\right)$.*

Proof Suppose that $i_0(f, g) > n_k\overline{b_k}$ and let $g = g_1 \cdots g_s$ with irreducible $g_j \in \mathbf{K}[[x, y]]$, for $j \in \{1, \ldots, s\}$. Then there exists $j \in \{1, \ldots, s\}$ such that

$$\frac{i_0(f, g_j)}{i_0(g_j, x)} > \frac{e_{k-1}\overline{b_k}}{n}. \tag{15}$$

Indeed, suppose that inequality (15) is not true. Then $i_0(f, g_j) \leq \frac{e_{k-1}\overline{b_k}}{n} i_0(g_j, x)$ for all $j \in \{1, \ldots, s\}$ and we get $i_0(f, g) = \sum_{j=1}^{s} i_0(f, g_j) \leq \sum_{j=1}^{s} \frac{e_{k-1}\overline{b_k}}{n} i_0(g_j, x) = \frac{e_{k-1}\overline{b_k}}{n} i_0(g, x) = n_k\overline{b_k}$ which contradicts the assumption about $i_0(f, g)$. The inequality (15) implies by Lemma 5.6 that $i_0(g_j, x) = q\frac{n}{e_k}$ for some integer $q > 0$. On the other hand $i_0(g_j, x) \leq i_0(g, x) = \frac{n}{e_k}$. Therefore $q = 1$ and $i_0(g_j, x) = i_0(g, x)$. Recall that g_j divides g, g_j is irreducible and ord $g_j(0, y) =$ ord $g(0, y)$, thus g_j is associated with g, which proves the irreducibility of g. We get $\overline{\text{char}_x} g = \left(\frac{\overline{b_0}}{e_k}, \ldots, \frac{\overline{b_k}}{e_k}\right)$ from the second part of Lemma 5.6. □

Corollary 5.8 (Abhyankar–Moh irreducibility criterion) *If $i_0(g, x) = n$ and $i_0(f, g) > n_h\overline{b_h}$ then g is irreducible and $\overline{\text{char}_x} g = \overline{\text{char}_x} f$.*

Notes

The Abhyankar–Moh irreducibility criterion was proved in [3,4] (Lemma 3.4) and explained in details in [2] (Theorem 12.4). The original version of the criterion was given for meromorphic curves. Using Puiseux series the authors had to assume $n \not\equiv 0$ (mod char $\mathbf{K}$). The version of the criterion presented in this paper is borrowed from [25] where the result is proved for the case char $\mathbf{K} = 0$.

6 Characterization of the semigroups associated with branches

In this section we give a new proof of the well-known theorem on the existence of branches with given semigroup (see [7] and [10]). Following Teissier [44] we give explicitly the equation of a plane curve with given characteristic. Our proof is written in the spirit of this paper, we do not use the technique of deformations. Here is the main result of this section.

Theorem 6.1 *Let $(\overline{b_0}, \ldots, \overline{b_h})$ be an n-characteristic sequence. Suppose there exists a distinguished irreducible polynomial $f_{h-1} \in \mathbf{K}[[x]][y]$ such that $\overline{\mathrm{char}}_x f_{h-1} = \left(\frac{\overline{b_0}}{e_{h-1}}, \ldots, \frac{\overline{b_{h-1}}}{e_{h-1}}\right)$. Let $f_0, \ldots, f_{h-2} \in \mathbf{K}[[x]][y]$ be a sequence of key polynomials of f_{h-1}. Let $a_0, \ldots, a_{h-1}$ be the (unique) sequence of integers such that $a_0\overline{b_0} + a_1\overline{b_1} + \cdots + a_{h-1}\overline{b_{h-1}} = n_h\overline{b_h}$, where $0 < a_0$ and $0 \leq a_i < n_i$ for $i \in \{1, \ldots, h-1\}$ and let $c \in \mathbf{K}\backslash\{0\}$. Put $f_h = f_{h-1}^{n_h} + cx^{a_0} f_0^{a_1} \cdots f_{h-2}^{a_{h-1}}$. Then*

1. *f_h is a distinguished irreducible polynomial of degree n, that is, $i_0(f_h, x) = \deg_y f_h$,*
2. *$\overline{\mathrm{char}}_x f_h = (\overline{b_0}, \ldots, \overline{b_h})$ and $f_0, \ldots, f_{h-1}$ are key polynomials of f_h.*

Proof Since f_{h-1} is a distinguished polynomial of degree $\frac{n}{e_{h-1}}$ and $a_0 > 0$, we have

$$\begin{aligned} i_0(f_h, x) &= i_0(f_{h-1}^{n_h} + cx^{a_0} f_0^{a_1} \cdots f_{h-2}^{a_{h-1}}, x) = i_0(f_{h-1}^{n_h}, x) \\ &= n_h i_0(f_{h-1}, x) = n_h \frac{n}{e_{h-1}} = n. \end{aligned}$$

To calculate $\deg_y f_h$ observe that $\deg_y f_{h-1}^{n_h} = n_h \deg_y f_{h-1} = n_h \frac{n}{e_{h-1}} = n$ and $\deg_y cx^{a_0} f_0^{a_1} \cdots f_{h-2}^{a_{h-1}} = a_1 \frac{n}{e_0} + \cdots + a_{h-1} \frac{n}{e_{h-2}} \leq (n_1 - 1)\frac{n}{e_0} + \cdots + (n_{h-1} - 1)\frac{n}{e_{h-2}} = \frac{n}{e_{h-1}} - 1 < n$. Therefore we get $\deg_y f_h = n$. The proof that f_h is irreducible is harder. We need auxiliary lemmas. □

Lemma 6.2 $i_0(f_h, f_{h-1}) = \overline{b_h}$.

Proof

$$\begin{aligned} i_0(f_h, f_{h-1}) &= i_0(f_{h-1}^{n_h} + cx^{a_0} f_0^{a_1} \cdots f_{h-2}^{a_{h-1}}, f_{h-1}) = i_0(x^{a_0} f_0^{a_1} \cdots f_{h-2}^{a_{h-1}}, f_{h-1}) \\ &= a_0 i_0(x, f_{h-1}) + a_1 i_0(f_0, f_{h-1}) + \cdots + a_{h-1} i_0(f_{h-2}, f_{h-1}) \\ &= a_0 \frac{\overline{b_0}}{e_{h-1}} + a_1 \frac{\overline{b_1}}{e_{h-1}} + \cdots + a_{h-1} \frac{\overline{b_{h-1}}}{e_{h-1}} = \frac{1}{e_{h-1}} n_h \overline{b_h} = \overline{b_h}. \end{aligned}$$

□

Lemma 6.3 *There exists an irreducible factor ϕ of f_h such that*

$$\frac{i_0(f_{h-1},\phi)}{i_0(\phi,x)} > \frac{n_{h-1}\overline{b_{h-1}}}{n}.$$

Proof Let $f_h = \phi_1 \cdots \phi_s$ with irreducible factors $\phi_i \in \mathbf{K}[[x, y]]$ for $i \in \{1, \dots, s\}$. Suppose that $\frac{i_0(f_{h-1},\phi_i)}{i_0(\phi_i,x)} \leq \frac{n_{h-1}\overline{b_{h-1}}}{n}$ for all $i \in \{1, \dots, s\}$. By Lemma 6.2 we get

$$\begin{aligned}\overline{b_h} &= i_0(f_{h-1}, f_h) = \sum_{i=1}^{s} i_0(f_{h-1},\phi_i) \leq \sum_{i=1}^{s} \frac{n_{h-1}\overline{b_{h-1}}}{n} i_0(\phi_i, x)\\ &= \frac{n_{h-1}\overline{b_{h-1}}}{n}\sum_{i=1}^{s} i_0(\phi_i, x) = \frac{n_{h-1}\overline{b_{h-1}}}{n} i_0(f_h, x) = n_{h-1}\overline{b_{h-1}} < \overline{b_h},\end{aligned}$$

which is a contradiction. □

Lemma 6.4 *Let ϕ be an irreducible factor of f_h such that $\frac{i_0(f_{h-1},\phi)}{i_0(\phi,x)} > \frac{n_{h-1}\overline{b_{h-1}}}{n}$. Then there exists $\nu \in \{1, \dots, n_h\}$ such that $i_0(\phi, x) = \nu\frac{\overline{b_0}}{e_{h-1}}$ and $i_0(\phi, f_k) = \nu\frac{\overline{b_{k+1}}}{e_{h-1}}$ for $k < h - 1$.*

Proof Recall that $\overline{\operatorname{char}_x} f_{h-1} = \left(\frac{\overline{b_0}}{e_{h-1}}, \dots, \frac{\overline{b_{h-1}}}{e_{h-1}}\right)$. Applying Lemma 5.6 to the irreducible power series f_{h-1} and ϕ (note that $\frac{n_{h-1}\overline{b_{h-1}}}{n} = \frac{n_{h-1}\frac{\overline{b_{h-1}}}{e_{h-1}}}{\frac{n}{e_{h-1}}}$) we conclude that $i_0(\phi, x) \equiv 0 \left(\operatorname{mod} \frac{n}{e_{h-1}}\right)$. Therefore we can write $i_0(\phi, x) = \nu\frac{n}{e_{h-1}}$ with $\nu \leq e_{h-1} = n_h$ since $i_0(\phi, x) \leq i_0(f_h, x) = n$. Fix $k < h - 1$ and consider the three branches $\{f_k = 0\}$, $\{f_{h-1} = 0\}$ and $\{\phi = 0\}$. We get $d_x(f_k,\phi) = \frac{e_{h-1}e_k i_0(f_k,\phi)}{\nu n^2}$, $d_x(f_{h-1},\phi) = \frac{i_0(f_{h-1},\phi)}{i_0(\phi,x)\frac{n}{e_{h-1}}} > \frac{n_{h-1}\overline{b_{h-1}}}{n}\frac{e_{h-1}}{n} = \frac{e_{h-2}\overline{b_{h-1}}}{n^2}$, and $d_x(f_{h-1}, f_k) = \frac{\overline{b_{k+1}}/e_{h-1}}{(n/e_{h-1})(n/e_k)} = \frac{e_k\overline{b_{k+1}}}{n^2} \leq \frac{e_{h-2}\overline{b_{h-1}}}{n^2}$, for $k < h - 1$. Therefore $d_x(f_{h-1}, f_k) < d_x(f_{h-1},\phi)$ and by the STI we get $d_x(f_{h-1}, f_k) = d_x(f_k,\phi)$, which implies $i_0(f_k,\phi) = \nu\frac{\overline{b_{k+1}}}{e_{h-1}}$. □

Now we are in a position to check that f_h is an irreducible power series. Let ϕ be an irreducible factor of f_h such that in Lemma 6.3. Since $f_h = f_{h-1}^{n_h} + cx^{a_0} f_0^{a_1} \cdots f_{h-2}^{a_{h-1}}$ and ϕ is an irreducible factor of f_h we get $i_0(f_{h-1}^{n_h},\phi) = i_0(x^{a_0} f_0^{a_1} \cdots f_{h-2}^{a_{h-1}},\phi)$. Therefore, by Lemma 6.4 we have

$$\begin{aligned}n_h i_0(f_{h-1},\phi) &= a_0 i_0(x,\phi) + a_1 i_0(f_0,\phi) + \cdots + a_{h-1} i_0(f_{h-2},\phi)\\ &= a_0\nu\frac{\overline{b_0}}{e_{h-1}} + a_1\nu\frac{\overline{b_1}}{e_{h-1}} + \cdots + a_{h-1}\nu\frac{\overline{b_{h-1}}}{e_{h-1}} = \frac{\nu}{e_{h-1}} n_h\overline{b_h} = \nu\overline{b_h}.\end{aligned}$$

Since $\nu\overline{b_h} \equiv 0 \pmod{n_h}$ and $\overline{b_h}$, $n_h = e_{h-1}$ are coprime we get $\nu \equiv 0 \pmod{n_h}$ and $\nu = n_h$ because $1 \leq \nu \leq n_h$.

From Lemma 6.4 we get $i_0(\phi, x) = n_h \frac{\overline{b_0}}{e_{h-1}} = \overline{b_0} = n = i_0(f_h, x)$. Since ϕ divides f_h we get $f_h = \phi\psi$ in $\mathbf{K}[[x, y]]$ with $\psi(0) \neq 0$. Therefore f_h is irreducible.

Now we prove the second statement of the theorem. First we check that $i_0(f_h, f_k) = \overline{b_{k+1}}$ for $k \in \{0, 1, \ldots, h-1\}$. We have $i_0(f_h, f_{h-1}) = \overline{b_h}$ by Lemma 6.2. Therefore we may assume that $h > 1$ and $k < h-1$. Applying Lemma 6.4 to the power series $\phi = f_h$ we get $i_0(f_h, f_k) = n_h \frac{\overline{b_{k+1}}}{e_{h-1}} = \overline{b_{k+1}}$ since $\nu = n_h$. Recall that $\deg_y f_k = \frac{n}{e_k}$ for $k \in \{0, 1, \ldots, h-1\}$. Using Lemma 4.1 we conclude that $\overline{\mathrm{char}_x}\, f_h = (\overline{b_0}, \ldots, \overline{b_h})$ and that $f_0, \ldots, f_{h-1}$ is a sequence of key polynomials of f_h. □

Theorem 6.5 (Bresinsky–Angermüller) *Let $\overline{b_0}, \ldots, \overline{b_h}$ be a sequence of positive integers. Then the following two conditions are equivalent:*

1. *There is an irreducible power series $f \in \mathbf{K}[[x]][y]$ such that $i_0(f, x) = \overline{b_0}$ and $\overline{b_0}, \ldots, \overline{b_h}$ is the $\overline{b_0}$-minimal sequence of generators of the semigroup $\Gamma(f)$.*
2. *The numbers $\overline{b_0}, \ldots, \overline{b_h}$ form a $\overline{b_0}$-characteristic sequence.*

Proof The implication (1) $\Longrightarrow$ (2) follows from the Semigroup Theorem (Theorem 3.2). To check that (2) $\Longrightarrow$ (1) we proceed by induction on the length h of the characteristic sequence using Theorem 6.1. If $h = 0$ then $(\overline{b_0}) = (1)$ and we take $f = y$. Let $h > 0$ and suppose that the implication (2) $\Longrightarrow$ (1) is true for $h-1$. Then there exists an irreducible distinguished polynomial $f_{h-1} \in \mathbf{K}[[x]][y]$ such that $\Gamma(f_{h-1}) = \mathbf{N}\frac{\overline{b_0}}{e_{h-1}} + \cdots + \mathbf{N}\frac{\overline{b_{h-1}}}{e_{h-1}}$. Let $f_0, \ldots, f_{h-2}$ be a sequence of key polynomials of f_{h-1}. Take $f = f_{h-1}^{n_h} + x^{a_0} f_0^{a_1} \cdots f_{h-2}^{a_{h-1}}$, where $0 < a_0$ and $0 \leq a_i < n_i$ for $i \in \{1, \ldots, h-1\}$ is the (unique) sequence of integers such that $a_0\overline{b_0} + a_1\overline{b_1} + \cdots + a_{h-1}\overline{b_{h-1}} = n_h\overline{b_h}$. Then by Theorem 6.1 f is an irreducible power series and $\Gamma(f) = \mathbf{N}\overline{b_0} + \cdots + \mathbf{N}\overline{b_h}$. □

Let $(\overline{b_0}, \ldots, \overline{b_h})$ be an n-characteristic sequence. For any $k \in \{1, \ldots, h\}$ we have Bézout's relation $n_k\overline{b_k} = a_{k0}\overline{b_0} + a_{k1}\overline{b_1} + \cdots + a_{kk-1}\overline{b_{k-1}}$, where $a_{k0} > 0$ and $0 \leq a_{ki} < n_i$ for $i \in \{1, \ldots, k-1\}$. Take $c_1, \ldots, c_h \in \mathbf{K}\backslash\{0\}$ and define in a recurrent way the polynomials $g_0, \ldots, g_h$ by putting $g_0 = y$, $g_1 = g_0^{n_1} + c_1 x^{a_{10}} = y^{n/e_1} + c_1 x^{\overline{b_1}/e_1}, \ldots, g_h = g_{h-1}^{n_h} + c_h x^{a_{h0}} g_0^{a_{h1}} \cdots g_{h-2}^{a_{hh-1}}$.

Theorem 6.6 (cf. [44] and [37]) *The polynomials $g_0, \ldots, g_h$ are distinguished and irreducible. We have $\overline{\mathrm{char}_x}\, g_k = \left(\frac{\overline{b_0}}{e_k}, \ldots, \frac{\overline{b_h}}{e_k}\right)$. The sequence $g_0, \ldots g_{k-1}$ is a sequence of key polynomials of g_k.*

Proof The theorem follows from Theorem 6.1 by induction on k. □

Notes

Theorem 6.5 characterizing the semigroups associated with branches is due to Bresinsky [10] (the case of characteristic 0) and to Angermüller [7] (the case of arbitrary characteristic, see also [24]). Both authors consider only *generic* case, i.e. $i_0(f, x) = \mathrm{ord}\, f$. Theorem 6.6 which gives an explicit equation of the branch with given semigroup was obtained by Teissier by the method of deformations of the monomial curve associated with a branch. Another proof was given by Reguera López in [37].

7 Description of branches with given semigroup

We need two preliminary lemmas.

Lemma 7.1 *Let* $f \in \mathbf{K}[[x]][y]$ *be a distinguished irreducible polynomial of degree* $n > 0$. *Suppose that* $\overline{\mathrm{char}}_x f = (\overline{b_0}, \dots, \overline{b_h})$, $\overline{b_0} = n$ *and let* $f_0, f_1, \dots, f_{h-1}$ *be a sequence of key polynomials of* f. *Then any polynomial* $g \in \mathbf{K}[[x]][y]$ *of* y*-degree strictly less than* n *has a unique expansion of the form*

$$g = \sum g_{\alpha_1,\dots,\alpha_h} f_0^{\alpha_1} \cdots f_{h-1}^{\alpha_h}, \quad g_{\alpha_1,\dots,\alpha_h} \in \mathbf{K}[[x]],$$

where $0 \le \alpha_1 < n_1$, ..., $0 \le \alpha_h < n_h$. *Moreover*

1. *the* y*-degrees of the terms appearing in the right-hand side of the preceding equality are all distinct,*
2. $i_0(f, g) = \inf\{(\mathrm{ord}\ g_{\alpha_1,\dots,\alpha_h})n + \alpha_1\overline{b_1} + \cdots + \alpha_h\overline{b_h} \ : \ 0 \le \alpha_i < n_i \text{ for } i = 1, \dots, h\}$.

Proof The existence and uniqueness of the expansion and the inequality for the degrees holds for polynomials with coefficients in arbitrary integral domain (see [2], Sect. 2). The formula for the intersection multiplicity follows from the observation that the intersection multiplicities $i_0(f, g_{\alpha_1,\dots,\alpha_h} f_0^{\alpha_1} \cdots f_{h-1}^{\alpha_h}) = (\mathrm{ord}\ g_{\alpha_1,\dots,\alpha_h})n + \alpha_1\overline{b_1} + \cdots + \alpha_h\overline{b_h}$ are pairwise distinct by the uniqueness of Bézout's relation. □

Lemma 7.2 *Under the notation and assumptions introduced above, if* $\deg_y g < n/e_k$ *then* $i_0(f, g) = e_k i_0(f_k, g)$.

Proof Suppose that $\deg_y g < n/e_k$. Then by Lemma 7.1 we get

$$g = \sum g_{\alpha_1,\dots,\alpha_h} f_0^{\alpha_1} \cdots f_{h-1}^{\alpha_h},$$

where $0 \le \alpha_i < n_i$, for $i \in \{1, \dots, h\}$. Since $\deg_y g < n/e_k$ we have, by the first statement of Lemma 7.1, $\alpha_{k+1} = \cdots = \alpha_h = 0$.

By Proposition 4.2 f_k is an irreducible distinguished polynomial, $\overline{\mathrm{char}}_x f_k = \left(\frac{\overline{b_0}}{e_k}, \dots, \frac{\overline{b_k}}{e_k}\right)$ and $f_0, \dots, f_{k-1}$ are key polynomials of f_k. Therefore there exist $\alpha_1, \dots, \alpha_k$ such that

$$i_0(f_k, g) = (\mathrm{ord}\ g_{\alpha_1,\dots,\alpha_k,0,\dots,0})\frac{n}{e_k} + \alpha_1\frac{\overline{b_1}}{e_k} + \cdots + \alpha_k\frac{\overline{b_k}}{e_k} = \frac{1}{e_k} i_0(f, g)$$

and the lemma follows. □

Let $\phi, f \in \mathbf{K}[[x]][y]$ be distinguished polynomials such that $N = \frac{\deg_y f}{\deg_y \phi}$ is a positive integer. Consider the ϕ-adic expansion of f:

$$f = \phi^N + \alpha_1\phi^{N-1} + \cdots + \alpha_N, \quad \deg_y \alpha_i < \deg_y \phi \text{ for } i \in \{1, \dots, N\}.$$

Put $\alpha_0 = 1$ and $I = \{i \in [0, N] \ :\ i_0(\alpha_i, \phi) \neq +\infty\}$. We define the *Newton polygon* $\Delta_{x,\phi}(f)$ of f with respect to the pair (x, ϕ) by setting

$$\Delta_{x,\phi}(f) = \text{convex} \bigcup_{i \in I} \left\{(i_0(\alpha_i, \phi), N - i) + \mathbf{R}^2_{\geq 0}\right\}.$$

The polygon $\Delta_{x,\phi}(f)$ intersects the vertical axis in the point $(0, N)$ and the horizontal axis in the point $(i_0(\alpha_N, \phi), 0) = (i_0(f, \phi), 0)$ provided that $i_0(f, \phi) \neq +\infty$. If $\phi = y$ then $\Delta_{x,\phi}(f) = \Delta_{x,y}(f)$ is the usual Newton polygon of f in coordinates (x, y).

In the sequel we use Teissier's notation (see [45]): for any integers $k, l > 0$ we put

$$\left\{\frac{k}{\overline{l}}\right\} = \text{convex}\{((k, 0) + \mathbf{R}^2_{\geq 0}) \cup ((0, l) + \mathbf{R}^2_{\geq 0})\}.$$

Proposition 7.3 *If $f \in \mathbf{K}[[x, y]]$ is an irreducible power series, $\overline{\text{char}}_x f = (\overline{b_0}, \ldots, \overline{b_h})$ and $f_0, f_1, \ldots, f_h$ is a sequence of key polynomials of f then*

$$\Delta_{x, f_{k-1}}(f_k) = \left\{\frac{\overline{b_k}/e_k}{\overline{e_{k-1}/e_k}}\right\}.$$

Proof The f_{k-1}-adic expansion of f_k is of the form $f_k = f_{k-1}^{n_k} + a_1 f_{k-1}^{n_k - 1} + \cdots + a_{n_k}$, where $\deg_y a_i < \frac{n}{e_{k-1}}$ for $i \in \{1, \ldots, n_k\}$. By Proposition 4.3 we have $i_0(f, a_i) > i\overline{b_k}$ for $0 < i < n_k$ and $i_0(f, a_{n_k}) = n_k \overline{b_k}$. By Lemma 7.2 we get $i_0(f, a_i) = e_{k-1} i_0(f_{k-1}, a_i)$ for $0 < i \leq n_k$. Therefore we have $i_0(f_{k-1}, a_i) > i \frac{\overline{b_k}}{e_{k-1}}$ for $0 < i < n_k$ and $i_0(f_{k-1}, a_{n_k}) = \frac{\overline{b_k}}{e_k}$, which implies $\Delta_{x, f_{k-1}}(f_k) = \left\{\frac{\overline{b_k}/e_k}{\overline{e_{k-1}/e_k}}\right\}$. □

Proposition 7.4 *Let f be an irreducible distinguished polynomial of degree $n > 1$. Let $\overline{\text{char}}_x f = (\overline{b_0}, \ldots, \overline{b_h})$ and let $\phi \in \mathbf{K}[[x]][y]$ be an $(h-1)$-key polynomial of f. Then*

1. $\overline{\text{char}}_x \phi = \left(\frac{\overline{b_0}}{e_{h-1}}, \ldots, \frac{\overline{b_{h-1}}}{e_{h-1}}\right)$,
2. $\Delta_{x,\phi}(f) = \left\{\frac{\overline{b_h}}{\overline{e_{h-1}}}\right\}$.

Proof The proposition follows from Propositions 4.2 and 7.3. □

The following theorem is the main result of this section.

Theorem 7.5 *Let $(\overline{b_0}, \ldots, \overline{b_h})$ be an n-characteristic sequence ($n > 1$). Let $f \in \mathbf{K}[[x]][y]$ be a distinguished polynomial of degree n for which there exists an irreducible distinguished polynomial $\phi \in \mathbf{K}[[x]][y]$ such that*

1. $\overline{\text{char}}_x \phi = \left(\frac{\overline{b_0}}{e_{h-1}}, \ldots, \frac{\overline{b_{h-1}}}{e_{h-1}}\right)$,
2. $\Delta_{x,\phi}(f) = \left\{\frac{\overline{b_h}}{\overline{e_{h-1}}}\right\}$.

Then f is irreducible, $\overline{\mathrm{char}}_x f = (\overline{b_0}, \ldots, \overline{b_h})$ and ϕ is an $(h-1)$- key polynomial of f.

Proof Let $\phi_0, \ldots, \phi_{h-2}, \phi_{h-1} = \phi$ be a sequence of key polynomials of ϕ.

From the assumption about $\Delta_{x,\phi}(f)$ it follows that if $f = \phi^{n_h} + \alpha_1 \phi^{n_h - 1} + \cdots + \alpha_{n_h}$, $\deg_y \alpha_i < \frac{n}{e_{h-1}}$ for $i \in \{1, \ldots, n_h\}$ is the ϕ-adic expansion of f then

$$(1)\ \ i_0(\alpha_{n_h}, \phi) = i_0(f, \phi) = \overline{b_h},$$

and

$$(2)\ \ i_0(\alpha_i, \phi) > i\frac{\overline{b_h}}{n_h} \ \text{ for } 0 < i < n_h$$

(note that $\gcd(n_h, \overline{b_h}) = \gcd(e_{h-1}, \overline{b_h}) = e_h = 1$ whence the strict inequality in (2)).

There exists a unique sentence of integers $l_0, \ldots, l_{h-1}$ such that $l_0\overline{b_0} + \cdots + l_{h-1}\overline{b_{h-1}} = e_{h-1}\overline{b_h}$, where $l_0 > 0$ and $0 \le l_i < n_i$ for $i \in \{1, \ldots, h-1\}$. Therefore we have $i_0(\phi, \alpha_{n_h}) = \overline{b_h} = i_0(\phi, x^{l_0}\phi_0^{l_1} \cdots \phi_{h-2}^{l_{h-1}})$. Let $c \in \mathbf{K}$ be a constant such that $i_0(\phi, \alpha_{n_h} - cx^{l_0}\phi_0^{l_1} \cdots \phi_{h-2}^{l_{h-1}}) > i_0(\phi, \alpha_{n_h}) = \overline{b_h}$. Put $\tilde{f} = \phi^{n_h} + cx^{l_0}\phi_0^{l_1} \cdots \phi_{h-2}^{l_{h-1}}$. Then by Theorem 6.1 $\tilde{f} \in \mathbf{K}[[x]][y]$ is an irreducible distinguished polynomial of degree n, $\overline{\mathrm{char}}_x \tilde{f} = (\overline{b_0}, \ldots, \overline{b_h})$ and ϕ is a key polynomial of degree $\frac{n}{e_{h-1}}$ of $\tilde{f}$.

We have $i_0(f, x) = i_0(\tilde{f}, x) = n$. Let $\tilde{\alpha}_{n_h} = \alpha_{n_h} - cx^{l_0}\phi_0^{l_1} \cdots \phi_{h-2}^{l_{h-1}}$ and consider

$$\begin{aligned}(3)\ \ i_0(\tilde{f}, f) &= i_0(\phi^{n_h} + cx^{l_0}\phi_0^{l_1} \cdots \phi_{h-2}^{l_{h-1}}, \phi^{n_h} + \alpha_1\phi^{n_h-1} + \cdots + \alpha_{n_h}) \\ &= i_0(\tilde{f}, \alpha_1\phi^{n_h-1} + \cdots + \alpha_{n_{h-1}}\phi + \tilde{\alpha}_{n_h}) \\ &\ge \inf\{i_0(\tilde{f}, \alpha_1\phi^{n_h-1}), \ldots, i_0(\tilde{f}, \alpha_{n_h-1}\phi), i_0(\tilde{f}, \tilde{\alpha}_{n_h})\}\end{aligned}$$

since $\tilde{f}$ is irreducible. Fix $i \in \{1, \ldots, n_h - 1\}$. Then

$$(4)\ \ i_0(\tilde{f}, \alpha_i\phi^{n_h-i}) = i_0(\tilde{f}, \alpha_i) + (n_h - i)i_0(\tilde{f}, \phi) = e_{h-1}i_0(\phi, \alpha_i) + (n_h - i)\overline{b_h}$$

since ϕ is an $(h-1)$-th key polynomial of $\tilde{f}$ and $i_0(\tilde{f}, \alpha_i) = e_{h-1}i_0(\phi, \alpha_i)$ by Lemma 7.2. Using (2) and (3) we get

$$(5)\ \ i_0(\tilde{f}, \alpha_i\phi^{n_h-i}) > e_{h-1}i\frac{\overline{b_h}}{e_{h-1}} + (n_h - i)\overline{b_h} = n_h\overline{b_h}$$

for $0 < i < n_h$. Moreover, again by Lemma 7.2

$$(6)\ \ i_0(\tilde{f}, \tilde{\alpha}_{n_h}) = e_{h-1}i_0(\phi, \tilde{\alpha}_{n_h}) > e_{h-1}i_0(\phi, \alpha_{n_h}) = e_{h-1}\overline{b_h}.$$

Using (3), (5) and (6) we obtain $i_0(\tilde{f}, f) > e_{h-1}\overline{b_h}$ and the theorem follows from the Abhyankar–Moh irreducibility criterion (Corollary 5.8). □

Using Proposition 7.4 and Theorem 7.5 we get a recurrent description of the class of branches with given semigroup.

Theorem 7.6 *Let* $(\overline{b_0},\dots,\overline{b_h})$ *be an* n*-characteristic sequence* ($n>1$) *and let* $f\in\mathbf{K}[[x]][y]$ *be a distinguished polynomial of degree* n*. Then the following two conditions are equivalent*

1. f *is irreducible and* $\overline{\mathrm{char}_x} f=(\overline{b_0},\dots,\overline{b_h})$,
2. *there exists a distinguished irreducible polynomial* $\phi\in\mathbf{K}[[x]][y]$ *such that*
   (a) $\overline{\mathrm{char}_x}\phi=\left(\frac{\overline{b_0}}{e_{h-1}},\dots,\frac{\overline{b_{h-1}}}{e_{h-1}}\right)$,
   (b) $\Delta_{x,\phi}(f)=\left\{\frac{\overline{b_h}}{e_{h-1}}\right\}$.

To illustrate the above result let us write down

Corollary 7.7 *Let* $f\in\mathbf{K}[[x]][y]$ *be a distinguished polynomial of degree* $n>1$ *and let* $m>0$ *be an integer such that* $\gcd(n,m)=1$*. Then* f *is irreducible with* $\overline{\mathrm{char}_x} f=(n,m)$ *if and only if there exists a power series* $\psi(x)\in\mathbf{K}[[x]]$, $\psi(0)=0$ *such that*

$$f=(y+\psi(x))^n+\alpha_1(x)(y+\psi(x))^{n-1}+\cdots+\alpha_n(x),$$

where $\mathrm{ord}\,\alpha_i>i\frac{m}{n}$ *for* $0<i<n$ *and* $\mathrm{ord}\,\alpha_n=m$.

Theorem 7.8 (Abhyankar's irreducibility criterion) *Let* $f\in\mathbf{K}[[x]][y]$ *be a distinguished polynomial of degree* $n>1$*. Assume that* $n\not\equiv 0$ *(mod char* $\mathbf{K}$*). Then* f *is irreducible if and only if there exists an* n*-characteristic sequence* $\overline{b_0},\dots,\overline{b_h}$ *such that*

1. $i_0(f,\sqrt[e_{k-1}]{f})=\overline{b_k}$ *and*
2. $\Delta_{x,\sqrt[e_{k-1}]{f}}(\sqrt[e_k]{f})=\left\{\frac{\overline{b_k}/e_k}{e_{k-1}/e_k}\right\}$ *for* $k\in\{1,\dots,h\}$.

Proof The conditions are necessary: if f is irreducible and $\overline{\mathrm{char}_x} f=(\overline{b_0},\dots,\overline{b_h})$ then both statements hold by Theorem 4.5 and Proposition 7.3.

The conditions are sufficient: this assertion follows from Theorem 7.5 by induction on the length h of the n-characteristic sequence. □

To check the first condition we determine the sequences $\overline{b_0},\dots,\overline{b_h}$ and $e_0,\dots,e_h$ such that

- $\overline{b_0}=n$,
- $\overline{b_k}=i_0(f,\sqrt[e_{k-1}]{f})$, $e_k=\gcd(e_{k-1},\overline{b_k})$ for $k\in\{1,\dots,h\}$,
- $e_0>\cdots>e_h=\gcd(e_h,i_0(f,\sqrt[e_h]{f}))$.

The first condition holds if and only if $e_h=1$ and $n_{k-1}\overline{b_{k-1}}<\overline{b_k}$ for $k>1$.

Notes

The first description of the class of branches with given semigroup is due to Teissier [44] (see also [14] and [27]). Our approach is inspired by the papers by Abhyankar [1] and Kuo [28] (see also [31] and [8]). The generalization of the Newton polygon introduced by Kuo in [28] is useful in Valuation Theory [43], Section 5. Our presentation of Abhyankar's irreducibility criterion differs from the original one. Another version of Abhyankar's criterion is due to Cossart and Moreno-Socías [18] and [19] . A criterion of irreducibility based on different ideas was given recently by [23]. The g-adic expansions of polynomials and Newton polygons were applied to generalize the classical Shönemann–Eisenstein irreducibility criterion in the early twentieth century (see [33]).

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