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ALGEBRAIC GEOMETRY

A Separation Condition for Polynomial Mappings

by

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Summary. Some estimates of the Łojasiewicz exponent at infinity for a class of polynomial mappings are given.

1. A separation condition at infinity. Let $F = (F_1, \dots, F_n) : \mathbb{C}^n \to \mathbb{C}^n$ be a dominating polynomial mapping and let $G : \mathbb{C}^n \to \mathbb{C}^n$ be a non-constant polynomial. For every $z = (z_1, \dots, z_n) \in \mathbb{C}^n$ we put $|z| = \max\{|z_j| : j = 1, \dots, n\}$. We say that F and G are *separated at infinity* if there are constants $C, R > 0$, and $q \in \mathbb{R}$ such that

$$|F(z)| \geqslant C|G(z)|^q \quad \text{for } |G(z)| \geqslant R.$$

PROPOSITION 1.1. *The following three conditions are equivalent:*
(1) *F and G are separated at infinity,*
(2) $\{0\} \times \mathbb{C} \not\subset \overline{(F,G)(\mathbb{C}^n)}$,
(3) *there is a polynomial $P : \mathbb{C}^n \times \mathbb{C} \to \mathbb{C}$ such that $P(F,G) = 0$ and $P \mid \{0\} \times \mathbb{C} \neq 0$.*

P r o o f. Let $V = \overline{(F,G)(\mathbb{C}^n)}$. Obviously, V is an irreducible algebraic subset of $\mathbb{C}^n \times \mathbb{C}$. Moreover, $\dim V = n$ because F is dominating.

Hence there exists an irreducible polynomial $P_0 : \mathbb{C}^n \times \mathbb{C} \to \mathbb{C}$ such that $V = P_0^{-1}(0)$. Clearly $\{0\} \times \mathbb{C} \not\subset V$ if and only if $P_0 \mid \{0\} \times \mathbb{C} \neq 0$. To see that our three conditions are equivalent it suffices to observe that for any polynomial $P : \mathbb{C}^n \times \mathbb{C} \to \mathbb{C}$ we have $P \mid \{0\} \times \mathbb{C} \neq 0$ if and only if

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$P(w,t)=0$ implies that $C|t|^q \leqslant |w|$ for some constants $C>0, q\in\mathbb{R}$, and large $|t|$ (cf. [8], Lemma 3.1). □

Let us consider an example.

Example 1.2. Let $F:\mathbb{C}^3\to\mathbb{C}^3$ be given by the formula

$$F(x,y,z)=(xy+z, x(1+(xy+z)x^2y), y(1+(xy+z)x^2y)).$$

Then for every polynomial $G:\mathbb{C}^3\to\mathbb{C}$ such that $\deg G=1$ the pair F,G is not separated at infinity. Indeed, let $P:\mathbb{C}^3\times\mathbb{C}\to\mathbb{C}$ be a polynomial such that $P(F,G)=0$. It suffices to check by Proposition 1.1(3) that $P\mid\{0\}\times\mathbb{C}=0$. Let us fix $a\in\mathbb{C}\setminus\{0\}$ and put

$$\varphi_a(t)=(a^{-2}t, -a^{-3}t^{-2}, a(1+t^{-1}))\quad \text{for } t\in\mathbb{C}\setminus\{0\}.$$

Then $F(\varphi_a(t))=(a,0,0)$ if $t\neq 0$ and we get $P(a,0,0,G(\varphi_a(t)))=0$ for $t\neq 0$. Since $G\neq 0$, $(G\circ\varphi_a)(\mathbb{C}\setminus\{0\})$ is dense in $\mathbb{C}$. Therefore $P(a,0,0,t)=0$ for all $t\in\mathbb{C}$. Passing to the limit when $a\to 0$ we get $P\mid\{0\}\times\mathbb{C}=0$.

Let ${}^hQ:\mathbb{C}\times\mathbb{C}^n\to\mathbb{C}$ be the homogenization of a polynomial $Q:\mathbb{C}^n\to\mathbb{C}$.

LEMMA 1.3. *Suppose that the system of equations ${}^hF_1=\ldots={}^hF_n={}^hG=0$ has no solutions on the hyperplane at infinity $z_0=0$. Then $F=(F_1,\ldots,F_n)$ and G are separated at infinity provided that $F^{-1}(0)$ is finite.*

Proof. The mapping $(F,G):\mathbb{C}^n\to\mathbb{C}^{n+1}$ is proper, therefore the set $(F,G)(\mathbb{C}^n)$ is algebraic and we have $\overline{(F,G)(\mathbb{C}^n)}=(F,G)(\mathbb{C}^n)$. It is easy to see that condition (2) of Proposition 1.1 means that the set $F^{-1}(0)$ is finite. □

Now, let $d(F)=(\mathbb{C}(Z):\mathbb{C}(F))$ be the *geometric degree* of F, where $Z=(Z_1,\ldots,Z_n)$ (cf. [6], p. 40). If the fiber $F^{-1}(w)$ is finite we put

$$\delta_w(F)=d(F)-\sum_{z\in F^{-1}(w)}\operatorname{mult}_z F,$$

where $\operatorname{mult}_z F$ denotes the multiplicity of F at z. We have always (cf. [7, 8])

$$0\leqslant\delta_w(F)\leqslant\Big(\prod_{i=1}^n\deg F_i\Big)-\min_{i=1}^n(\deg F_i),$$

and $\delta_w(F)=0$ for generic $w\in\mathbb{C}^n$. Let us put $d(F,G)=(\mathbb{C}(Z):\mathbb{C}(F,G))$ for any polynomial G. We can state our main result

THEOREM 1.4. *Suppose that the fiber $F^{-1}(0)$ is finite and the pair F,G is separated at infinity. Then there are constants $C,R>0$ such that*

$$|F(z)|\geqslant C|G(z)|^{-\delta_0(F)/d(F,G)}\quad \textit{for } |G(z)|\geqslant R.$$

P r o o f. Let us keep the notation introduced in the proof of Proposition 1.1. According to ([8], Lemma 3.1), there are $C, R > 0$ such that if $P_0(w,t) = 0$ and $|t| \geqslant R$, then $C|t|^q \leqslant |w|$, where $q = \deg_T P_0(0,T) - \deg_T P_0(W,T)$. We have $P_0(F,G) = 0$ and P_0 is irreducible, therefore

$$\deg_T P_0(W,T) = (\mathbb{C}(F,G) : \mathbb{C}(F)) = d(F)/d(F,G)$$

and it suffices to check that

$$\deg_T P_0(0,T) \geqslant \sum_{z \in F^{-1}(0)} (\operatorname{mult}_z F)/d(F,G).$$

Let us assume that $\mu = \sum_{z \in F^{-1}(0)} \operatorname{mult}_z F \neq 0$, and let W be an open neighbourhood of the set $V \cap \{0\} \times \mathbb{C}$. There exist open neighbourhoods D of 0 and U of $F^{-1}(0)$ such that:

(a) $F|U : U \to D$ is an analytic μ-sheeted branched covering,
(b) $(F,G)(U) \subset W$.

Shrinking the neighbourhood W, if necessary, we can assume that all the fibres of the mapping $V \cap W \ni (w,t) \to w \in D$ have no more than $\deg_T P_0(0,T)$ points. For generic $a \in V$ we have $\#(F,G)^{-1}(a) = d(F,G)$. This implies $\mu \leqslant d(F,G) \deg_T P_0(0,T)$ and the proof is complete. □

Let us note here that $\delta_0(F) = 0$ if and only if there are constants $C, R > 0$ such that $|F(z)| \geqslant C$ for $|z| \geqslant R$ (cf. [3]). Therefore, Theorem 1.4 is interesting only if $\delta_0(F) > 0$. Below we present some applications of this theorem to the polynomial mappings.

2. Convenient mappings. Let $F : \mathbb{C}^n \to \mathbb{C}^n$ be a dominating polynomial mapping. We say that F is *convenient* if there is a basis $L_1, \ldots, L_n$ of the space $(\mathbb{C}^n)^*$ of linear forms on $\mathbb{C}^n$, such that F, L_i are separated at infinity for $i = 1, \ldots, n$.

PROPERTY 2.1. *If $F : \mathbb{C}^n \to \mathbb{C}^n$ is convenient, then there is an open neighbourhood V of $0 \in \mathbb{C}^n$ such that $F^{-1}(w)$ is finite for all $w \in V$.*

P r o o f. Choose polynomials P_i such that $P_i(F, L_i) = 0$ and $P_i \mid \{0\} \times \mathbb{C} \neq 0$ for $i = 1, \ldots, n$. There is a neighbourhood V of $0 \in \mathbb{C}^n$ such that $P_i \mid \{w\} \times \mathbb{C} \neq 0$ for every $w \in V$. Let $L : \mathbb{C}^n \to \mathbb{C}^n$ be the linear automorphism given by $L = (L_1, \ldots, L_n)$. Obviously, $L(F^{-1}(w))$ is finite for $w \in V$. Consequently, $F^{-1}(w)$ is finite if $w \in V$. □

PROPERTY 2.2. *Let $F = (F_1, \ldots, F_n)$ be a dominating polynomial mapping such that the system of homogeneous equations ${}^hF_1 = \ldots = {}^hF_n = 0$ has a finite number of solutions in the projective space $\mathbb{P}^n(\mathbb{C})$. Then F is convenient.*

The proof follows easily from Lemma 1.3 and the definition of convenient mappings.

Using Theorem 1.4 we get the following sharper version of the main result of [8].

THEOREM 2.3. *If* $F = (F_1, \dots, F_n) : \mathbb{C}^n \to \mathbb{C}^n$ *is a convenient mapping, then there exist* $C, R > 0$ *such that*

$$|F(z)| \geqslant C|z|^{-\delta_0(F)} \quad \textit{for } |z| \geqslant R.$$

Let us consider two examples.

Example 2.4 (cf. [1]). Let $D > 1$ be an integer and let us put

$$F(z) = (z_1^D, z_1 - z_2^D, \dots, z_{n-2} - z_{n-1}^D, 1 - z_{n-1} z_n^{D-1}).$$

Here $d(F) = D^n - D^{n-1}$ and $F^{-1}(0) = \emptyset$. Consequently, $\delta_0(F) = D^n - D^{n-1}$. The system of equations ${}^hF_1 = \dots = {}^hF_n = 0$ has the only solution $(0 : 0 : \dots : 0 : 1)$ in $\mathbb{P}^n(\mathbb{C})$, therefore F is convenient and $|F(z)| \geqslant C|z|^{-\delta_0(F)} = C|z|^{D^{n-1} - D^n}$ for $C > 0$ and large $|z|$. By taking the restriction of F to the curve given by

$$\varphi(T) = \Big(\frac{1}{T^{D^{n-1}(D-1)}}, \dots, \frac{1}{T^{D(D-1)}}, \frac{1}{T^{D-1}}, T\Big)$$

one checks that $-\delta_0(F) = D^{n-1} - D^n$ is the biggest possible exponent in our estimate (*Łojasiewicz exponent*).

Example 2.5. Let

$$F(x, y, z) = (x, x^2 y, x y^{s-1} z + 1)$$

where $s > 1$ is an integer. We have $d(F) = 1$, $F^{-1}(0) = \emptyset$. Hence $\delta_0(F) = 1$. One checks that $|F(x, y, z)| \geqslant C|(x, y, z)|^{-s}$ for a constant $C > 0$ and large $|(x, y, z)|$. Moreover, $-s$ is the best exponent in this estimate and so $-\delta_0(F) = -1$ is not good. We see that the assumption that "F is convenient mapping" in Theorem 2.3 is essential.

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