UNIVERSITATIS IAGELLONICAE ACTA MATHEMATICA
doi: 10.4467/20843828AM.15.001.3727 FASCICULUS LII, 2015

ON THE ABHYANKAR–MOH INEQUALITY

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Abstract. In their fundamental paper on the embeddings of the line in the plane, Abhyankar and Moh proved an important inequality which can be stated in terms of the semigroup associated with the branch at infinity of a plane algebraic curve. In this note we study the semigroups of integers satisfying the Abhyankar–Moh inequality and describe such semigroups with the maximum conductor.

Introduction. In this note we study the semigroups of integers appearing in connection with the Abhyankar–Moh inequality which is the main tool in proving the famous embedding line theorem (see [**1**, Main theorem]). Since the Abhyankar–Moh inequality can be stated in terms of the semigroup associated with the branch at infinity of a plane algebraic curve, it is natural to consider the semigroups for which such an inequality holds.

Section 1 being devoted to preparatory lemmas, in Section 2 we study such semigroups (we call them *Abhyankar–Moh semigroups*). Then, in Section 3 we give a simplified proof of the Abhyankar–Moh Embedding Line Theorem.

In what follows we need some basic properties of semigroups of the naturals. A subset G of $\mathbf{N}$ is a *semigroup* if it contains 0 and it is closed under addition. Let G be a nonzero semigroup and let $n \in G$, $n > 0$. Then there exists a unique sequence $(v_0, \dots, v_h)$ such that $v_0 = n$, $v_k = \min(G \backslash v_0\mathbf{N} + \cdots + v_{k-1}\mathbf{N})$ for $1 \le k \le h$ and $G = v_0\mathbf{N} + \cdots + v_h\mathbf{N}$. We call the sequence $(v_0, \dots, v_h)$ the *n-minimal system of generators of* G. If $n = \min(G\backslash\{0\})$ then we say that

2010 *Mathematics Subject Classification.* Primary 14R10; Secondary 32S05.
Key words and phrases. Branches at infinity, semigroups, sequences of divisors, Abhyankar–Moh inequality.
The second-named author was partially supported by the Spanish Project MTM2012-36917-C03-01.

$(v_0, \ldots, v_h)$ is the *minimal system of generators* of G (see [**6**, Proposition 6.1]). Each minimal system of generators is an increasing sequence.

1. Characteristic sequences and strongly increasing semigroups. A sequence of positive integers $(\bar{b}_0, \ldots, \bar{b}_h)$ will be called a *characteristic sequence* if it satisfies the following two axioms

1. Set $e_k = \gcd(\bar{b}_0, \ldots, \bar{b}_k)$ for $0 \leq k \leq h$. Then $e_k < e_{k-1}$ for $1 \leq k \leq h$ and $e_h = 1$.
2. $e_{k-1}\bar{b}_k < e_k\bar{b}_{k+1}$ for $1 \leq k \leq h-1$.

In what follows we put $n_k = \frac{e_{k-1}}{e_k}$ for $1 \leq k \leq h$. Therefore, $n_k > 1$ for $1 \leq k \leq h$ and $n_h = e_{h-1}$. If $h = 0$ there is exactly one characteristic sequence $(\bar{b}_0) = (1)$. If $h = 1$ then the sequence $(\bar{b}_0, \bar{b}_1)$ is a characteristic sequence if and only if $\gcd(\bar{b}_0, \bar{b}_1) = 1$.

The second axiom is essential if and only if $h \geq 2$.

Lemma 1.1. *Let $(\bar{b}_0, \ldots, \bar{b}_h)$, $h \geq 2$ be a characteristic sequence. Then*

(i) $\bar{b}_1 < \cdots < \bar{b}_h$ *and* $\bar{b}_0 < \bar{b}_2$.
(ii) *Let $\bar{b}_1 < \bar{b}_0$. If $\bar{b}_0 \not\equiv 0 \pmod{\bar{b}_1}$ then $(\bar{b}_1, \bar{b}_0, \bar{b}_2, \ldots, \bar{b}_h)$ is a characteristic sequence. If $\bar{b}_0 \equiv 0 \pmod{\bar{b}_1}$ then $(\bar{b}_1, \bar{b}_2, \ldots, \bar{b}_h)$ is a characteristic sequence.*

Proof. The lemma follows directly from the definition of characteristic sequence. $\square$

Proposition 1.2. *Let $G = \bar{b}_0\mathbf{N} + \cdots + \bar{b}_h\mathbf{N}$, where $(\bar{b}_0, \ldots, \bar{b}_h)$ is a characteristic sequence. Then*

1. *the sequence $(\bar{b}_0, \ldots, \bar{b}_h)$ is the $\bar{b}_0$-minimal system of generators of the semigroup G.*
2. $\min(G \backslash \{0\}) = \min(\bar{b}_0, \bar{b}_1)$.
3. *The minimal system of generators of G is $(\bar{b}_0, \ldots, \bar{b}_h)$ if $\bar{b}_0 < \bar{b}_1$, $(\bar{b}_1, \bar{b}_0, \bar{b}_2, \ldots, \bar{b}_h)$ if $\bar{b}_0 > \bar{b}_1$ and $\bar{b}_0 \not\equiv 0 \pmod{\bar{b}_1}$ and $(\bar{b}_1, \bar{b}_2, \ldots, \bar{b}_h)$ if $\bar{b}_0 \equiv 0 \pmod{\bar{b}_1}$.*
4. *Let $c = \sum_{k=1}^{h}(n_k - 1)\bar{b}_k - \bar{b}_0 + 1$. Then c is the conductor of G, that is the smallest element of G such that all integers larger than or equal to it are in G.*

Proof. *1.* Fix k, $1 \le k \le h$ and let $\overline{b} = \min(G \backslash \overline{b}_0\mathbf{N} + \cdots + \overline{b}_{k-1}\mathbf{N})$. Then $\overline{b} = q_0\overline{b}_0 + \cdots + q_h\overline{b}_h$, where $q_0, \ldots, q_h$ are nonnegative integers and $q_l \neq 0$ for an $l \ge k$. Thus $\overline{b} \ge \overline{b}_l \ge \overline{b}_k$, since the sequence $(\overline{b}_1, \ldots, \overline{b}_h)$ is increasing by Lemma 1.1(i). On the other hand, $\overline{b}_k \not\equiv 0 \pmod{e_{k-1}}$ and consequently $\overline{b}_k \notin \overline{b}_0\mathbf{N} + \cdots + \overline{b}_{k-1}\mathbf{N}$, which implies $\overline{b}_k \ge \overline{b}$. Therefore we get $\overline{b} = \overline{b}_k$ and we are done.

2. From $G = \overline{b}_0\mathbf{N} + \cdots + \overline{b}_h\mathbf{N}$ it follows that $\min(G \backslash \{0\}) = \min(\overline{b}_0, \ldots, \overline{b}_h) = \min(\overline{b}_0, \overline{b}_1)$ by Lemma 1.1(i).

3. Follows from Lemma 1.1(ii) and Property *1.*

4. If $\overline{b}_0 < \overline{b}_1$ then the formula for the conductor is proved in [**6**, Proposition 7.7, Proposition 7.9]. If $\overline{b}_0 > \overline{b}_1$, by Property *3.* we have two cases. Denote by $(\overline{\beta}_0, \ldots, \overline{\beta}_g)$ the minimal system of G. Put $\epsilon_k = \gcd(\overline{\beta}_0, \ldots, \overline{\beta}_k)$ for $0 \le k \le g$ and $\nu_k = \frac{\epsilon_{k-1}}{\epsilon_k}$ for $1 \le k \le g$.

<u>Case 1</u>: $\overline{b}_0 \not\equiv 0 \pmod{\overline{b}_1}$. Then $(\overline{\beta}_0, \ldots, \overline{\beta}_g) = (\overline{b}_1, \overline{b}_0, \overline{b}_2, \ldots, \overline{b}_h)$, $(\epsilon_0, \ldots, \epsilon_g) = (\overline{b}_1, \epsilon_1, \ldots, \epsilon_g)$ and $(\nu_1, \ldots, \nu_g) = \left(\frac{\overline{b}_1}{e_1}, n_2, \ldots, n_h\right)$.

<u>Case 2</u>: $\overline{b}_0 \equiv 0 \pmod{\overline{b}_1}$. Then $(\overline{\beta}_0, \ldots, \overline{\beta}_g) = (\overline{b}_1, \overline{b}_2, \ldots, \overline{b}_h)$, $(\epsilon_0, \ldots, \epsilon_g) = (\overline{b}_1, \epsilon_1, \ldots, \epsilon_g)$ and $(\nu_1, \ldots, \nu_g) = \left(\frac{\overline{b}_1}{e_2}, n_3, \ldots, n_h\right)$.

In both cases we get

$$\sum_{k=1}^{h}(n_k - 1)\overline{b}_k - \overline{b}_0 + 1 = \sum_{k=1}^{g}(\nu_k - 1)\overline{\beta}_k - \overline{\beta}_0 + 1 = c.$$

□

Corollary 1.3. *Let $G \subset \mathbf{N}$ be a nonzero semigroup. Then the following two conditions are equivalent*

(a) *the minimal system of generators of G is a characteristic sequence.*
(b) *G is generated by a characteristic sequence.*

A semigroup $G \subseteq \mathbf{N}$ is a *strongly increasing (s.i.) semigroup* if $G \neq (0)$ and satisfies one of the equivalent conditions (a), (b). In [**2**] and [**6**] the authors define s.i. semigroups by condition (a).

In what follows we denote $(\overline{\beta}_0, \ldots, \overline{\beta}_g)$ the minimal system of generators of an s.i. semigroup G and $(\epsilon_0, \ldots, \epsilon_g)$ the associated sequence of divisors $\epsilon_k = \gcd(\overline{\beta}_0, \ldots, \overline{\beta}_k)$ for $0 \le k \le g$.

COROLLARY 1.4. *Let $G \subseteq \mathbf{N}$ be an s.i. semigroup with minimal system of generators $(\overline{\beta}_0, \dots, \overline{\beta}_g)$. Then the characteristic sequences of generators distinct from the minimal system are $(\overline{\beta}_1, \overline{\beta}_0, \overline{\beta}_2, \dots, \overline{\beta}_g)$ and $(l\overline{\beta}_0, \overline{\beta}_0, \overline{\beta}_1, \dots, \overline{\beta}_g)$, where l is an integer such that $1 < l < \overline{\beta}_1/\overline{\beta}_0$.*

Corollaries 1.3 and 1.4 follow directly from Proposition 1.2.

2. Abhyankar–Moh semigroups. A semigroup $G \subseteq \mathbf{N}$ will be called an *Abhyankar–Moh semigroup of degree* $n > 1$ if it is generated by a characteristic sequence $(\overline{b}_0, \overline{b}_1, \dots, \overline{b}_h)$, $\overline{b}_0 = n$, satisfying the Abhyankar–Moh inequality

$$\text{(AM)} \qquad e_{h-1}\overline{b}_h < n^2.$$

By Corollary 1.3, every Abhyankar–Moh semigroup is an s.i. semigroup.

PROPOSITION 2.1. *Let $G \subseteq \mathbf{N}$ be an s.i. semigroup with the minimal system of generators $(\overline{\beta}_0, \dots, \overline{\beta}_g)$. Then G is an Abhyankar–Moh semigroup of degree $n > 1$ if and only if $\epsilon_{g-1}\overline{\beta}_g < n^2$ and $n = \overline{\beta}_1$ or $n = l\overline{\beta}_0$, where l is an integer such that $1 < l < \overline{\beta}_1/\overline{\beta}_0$.*

PROOF. Let $G = \overline{b}_0\mathbf{N} + \cdots + \overline{b}_h\mathbf{N}$, where $(\overline{b}_0, \overline{b}_1, \dots, \overline{b}_h)$ is a characteristic sequence such that $e_{h-1}\overline{b}_h < n^2$, $n = \overline{b}_0$. The sequence $(e_0\overline{b}_1, \dots, e_{h-1}\overline{b}_h)$ being increasing, we have $e_0\overline{b}_1 < n^2$. Since $e_0 = \overline{b}_0 = n$ we get $\overline{b}_1 < n = \overline{b}_0$. By Corollary 1.4 $(\overline{b}_0, \overline{b}_1, \dots, \overline{b}_h) = (\overline{\beta}_1, \overline{\beta}_0, \dots, \overline{\beta}_g)$ and $n = \overline{\beta}_1$ or $(\overline{b}_0, \overline{b}_1, \dots, \overline{b}_h) = (l\overline{\beta}_0, \overline{\beta}_0, \dots, \overline{\beta}_g)$ and $n = l\overline{\beta}_0$, where $1 < l < \overline{\beta}_1/\overline{\beta}_0$. □

Our main result is the following

THEOREM 2.2. *Let G be an Abhyankar–Moh semigroup of degree $n > 1$ and let c be the conductor of G. Then $c \leq (n-1)(n-2)$. If G is generated by a characteristic sequence $(\overline{b}_0, \overline{b}_1, \dots, \overline{b}_h)$, $\overline{b}_0 = n$, satisfying* (AM) *then $c = (n-1)(n-2)$ if and only if $\overline{b}_k = \frac{n^2}{e_{k-1}} - e_k$ for $1 \leq k \leq h$.*

PROOF. Let $\delta_0 = n$ and $\delta_k = \dfrac{n^2}{e_{k-1}} - \overline{b}_k$ for $1 \leq k \leq h$. We have $\delta_k = \dfrac{n^2 - e_{k-1}\overline{b}_k}{e_{k-1}} \geq \dfrac{n^2 - e_{h-1}\overline{b}_h}{e_{k-1}} > 0$ and $\gcd(\delta_0, \dots, \delta_k) = e_k$ for $0 \leq k \leq h$. Let $\gamma = \sum_{k=1}^{h}(n_k - 1)\delta_k - \delta_0 + 1$. Since $\delta_k \geq e_k$ we get $\gamma \geq \sum_{k=1}^{h}(n_k - 1)e_k - e_0 + 1 = 0$ with equality if and only if $\delta_k = e_k$ for $0 \leq k \leq h$.

On the other hand

$$\begin{aligned}\gamma &= \sum_{k=1}^{h}(n_k-1)\left(\frac{n^2}{e_{k-1}}-\overline{b}_k\right)-n+1\\ &= \sum_{k=1}^{h}(n_k-1)\frac{n^2}{e_{k-1}}-n+1-\sum_{k=1}^{h}(n_k-1)\overline{b}_k\\ &= (n-1)^2-\sum_{k=1}^{h}(n_k-1)\overline{b}_k=(n-1)(n-2)-c\end{aligned}$$

since $c=\sum_{k=1}^{h}(n_k-1)\overline{b}_k-\overline{b}_0+1$ by Proposition 1.2.4.
Therefore $c\le(n-1)(n-2)$ since $\gamma\ge 0$ and $c=(n-1)(n-2)$ if and only if $\gamma=0$ which is equivalent to $\delta_k=e_k$, that is $\dfrac{n^2}{e_{k-1}}-\overline{b}_k=e_k$.

□

Let $n>1$ be an integer. A sequence of integers $(e_0,\dots,e_h)$ will be called a *sequence of divisors of* n if e_k divides e_{k-1} for $1\le k\le h$ and $n=e_0>e_1>\dots>e_{h-1}>e_h=1$.

Now, we can give a simple description of the Abhyankar–Moh semigroups of degree $n>1$ with $c=(n-1)(n-2)$. Observe that if $(e_0,\dots,e_h)$ is a sequence of divisors of $n>1$ then the sequence

$$\left(n,n-e_1,\frac{n^2}{e_1}-e_2,\dots,\frac{n^2}{e_{k-1}}-e_k,\dots,\frac{n^2}{e_{h-1}}-1\right) \tag{1}$$

is a characteristic sequence satisfying the Abhyankar–Moh inequality. Let $G(e_0,\dots,e_h)$ be the semigroup generated by the sequence (1).

Proposition 2.3. *A semigroup $G\subseteq\mathbf{N}$ is an Abhyankar–Moh semigroup of degree $n>1$ with $c=(n-1)(n-2)$ if and only if $G=G(e_0,\dots,e_h)$ where $(e_0,e_1,\dots,e_h)$ is a sequence of divisors of n.*

Proof. Let $\overline{b}_0=n$, $\overline{b}_k=\frac{n^2}{e_{k-1}}-e_k$ for $1\le k\le h$.
Then $G(e_0,e_1,\dots,e_h)=\overline{b}_0\mathbf{N}+\dots+\overline{b}_h\mathbf{N}$ and the conductor of $G(e_0,e_1,\dots,e_h)$ is equal to $\sum_{k=1}^{h}(n_k-1)\overline{b}_k-\overline{b}_0+1=\sum_{k=1}^{h}(n_k-1)\left(\frac{n^2}{e_{k-1}}-e_k\right)-n+1=(n-1)(n-2)$. Therefore $G(e_0,e_1,\dots,e_h)$ is an Abhyankar–Moh semigroup of degree n with $c=(n-1)(n-2)$. The converse follows directly from the second part of Theorem 2.2. □

Corollary 2.4. *Let G be an Abhyankar–Moh semigroup of degree $n>1$ with $c=(n-1)(n-2)$ and let $n'=\min(G\backslash\{0\})$. Then $n-n'$ divides n.*

PROOF. Let $G = G(e_0, \ldots, e_h)$. Then $n' = n - e_1$ by Proposition 2.3 and the corollary follows. $\square$

COROLLARY 2.5. *Let G be an Abhyankar–Moh semigroup of degree $n > 1$ with $c = (n-1)(n-2)$ and let $(\overline{\beta}_0, \overline{\beta}_1, \ldots, \overline{\beta}_g)$ be the minimal system of generators of the semigroup G. Then $n = \overline{\beta}_1$ or $n = 2\overline{\beta}_0$.*

If $n = \overline{\beta}_1$ then $G = G(n, \epsilon_1, \ldots, \epsilon_g)$.

If $n = 2\overline{\beta}_0$ then $G = G(n, \epsilon_0, \ldots, \epsilon_g)$.

PROOF. By Proposition 2.1 $n = \overline{\beta}_1$ or $n = l\overline{\beta}_0$. Suppose that $n \neq \overline{\beta_1}$. Then $n = l\overline{\beta_0}$ and by Corollary 2.4 $n - \overline{\beta_0}$ divides n, that is $n = k(n - \overline{\beta_0})$ for an integer $k > 0$. Thus we get $l = (l-1)k$ which implies $l = 2$. The equalities for G follow from Corollary 1.4. $\square$

REMARK 2.6. Suppose that $(\overline{b}_0, \overline{b}_1, \ldots, \overline{b}_h)$, $\overline{b}_0 = n$, is a characteristic sequence which satisfies the Abhyankar–Moh inequality. Then the sequence of positive integers $\delta_0 = \overline{b}_0 = n$, $\delta_k = \dfrac{n^2}{e_{k-1}} - \overline{b}_k$ for $1 \leq k \leq h$, has the following properties

(i) the sequence of divisors $\gcd(\delta_0, \ldots, \delta_k) = e_k$, $1 \leq k \leq h$, is strictly decreasing, $e_h = 1$.
(ii) $\delta_1 < \delta_0$ and $\delta_k < n_{k-1}\delta_{k-1}$ for $2 \leq k \leq h$, where $n_k = \frac{e_{k-1}}{e_k}$ for $1 \leq k \leq h$.

Note that $(\delta_0, \ldots, \delta_h)$ defined above is not a δ-sequence in the sense of [**7**], that is in general, the property $n_k\delta_k \in \delta_0\mathbf{N} + \cdots + \delta_{k-1}\mathbf{N}$ fails. For example $(\overline{b}_0, \overline{b}_1, \overline{b}_2) = (6, 2, 7)$ is a characteristic sequence with the Abhyankar–Moh property but $n_2\delta_2 = 2 \notin \delta_0\mathbf{N} + \delta_1\mathbf{N} = 6\mathbf{N} + 4\mathbf{N}$ (see [**3**]).

3. Plane curves with one branch at infinity. In this section we use freely the properties of plane projective curves (see [**8**]).

Let $\mathbf{K}$ be an algebraically closed field of arbitrary characteristic. A projective plane curve C defined over $\mathbf{K}$ *has one branch at infinity* if there is a line (line at infinity) intersecting C in only one point O, and C has only one branch centered at this point. In what follows we denote by n the degree of C, by n' the multiplicity of C at O and we put $d := \gcd(n, n')$.

We call C *permissible* if $d \not\equiv 0 \pmod{\operatorname{char} \mathbf{K}}$. Let γ be a projective plane branch with a center O. The semigroup of γ is the set of the intersection numbers $(\gamma, \gamma')_O$, where γ' runs over all local analytic curves centered at O.

The following theorem is basic for studying the plane curves with one branch at infinity.

Theorem 3.1 (Abhyankar–Moh inequality [**1**]). *Assume that C is a permissible curve of degree $n > 1$. Then the semigroup G_O of the unique branch at infinity of C is an Abhyankar–Moh semigroup of degree n.*

A simple proof of the above theorem is given in [**5**] for the fields of characteristic 0. The proof is easily adopted to the positive characteristic case by using the Abhyankar–Moh theorem on approximate roots in the form given in [**4**, Theorem 3.5].

An important application of Theorem 3.1 is

Theorem 3.2 (Abhyankar–Moh Embedding Line Theorem, [**1**], Main theorem). *Assume that C is a rational projective irreducible curve of degree $n > 1$ with one branch at infinity and such that the center of the branch at infinity O is the unique singular point of C. Suppose that C is permissible and let n' be the multiplicity of C at O. Then $n - n'$ divides n.*

Proof. By Theorem 3.1 the semigroup G_O of the branch at infinity is an Abhyankar–Moh semigroup of degree n. Let c be the conductor of the semigroup G_O. Using the Noether formula for the genus of projective plane curve we get $c = (n-1)(n-2)$. Then the theorem follows from Corollary 2.4. $\square$

Remark 3.3. Keep the assumptions and notations from the beginning of this section. Let $\overline{\beta}_0 = n'$, $\overline{\beta}_1, \cdots$ be the minimal system of generators of the semigroup G_O. From Proposition 2.1 if follows that the line at infinity L has maximal contact with C, that is intersects C with multiplicity $\overline{\beta}_1$ if and only if $n \not\equiv 0 \pmod{n'}$. Using the main result on the approximate roots (see [**4**, Theorem 3.5]) one proves that if $n \equiv 0 \pmod{n'}$) then there is an irreducible curve C' of degree n/n' intersecting C with multiplicity $\overline{\beta}_1$. In particular, if C is rational then by Corollary 2.5 we get $n/n' = 2$ (if $n \equiv 0 \pmod{n'}$) and C' is a nonsingular curve of degree 2.

Remark 3.3 gives the answer to a question raised by Bernard Teissier.

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Received February 18, 2015

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