Introducción a códigos polares

Eduardo Camps Moreno

Escuela Superior de Física y Matemáticas

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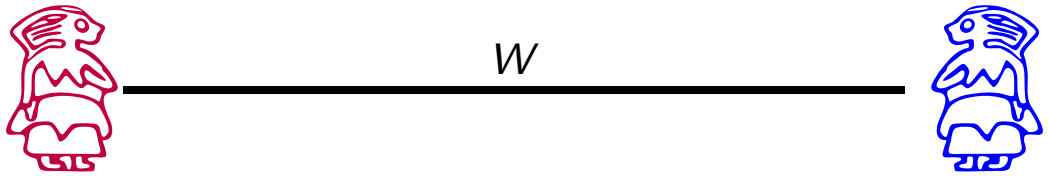
Universidad de la Laguna, Tenerife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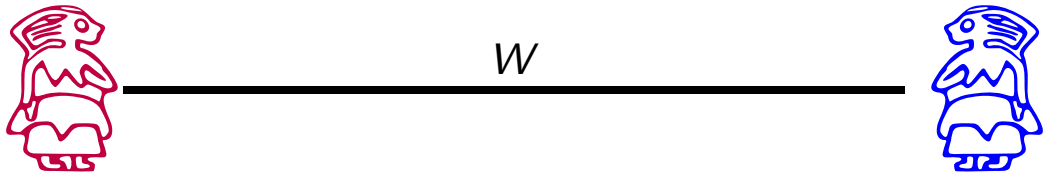
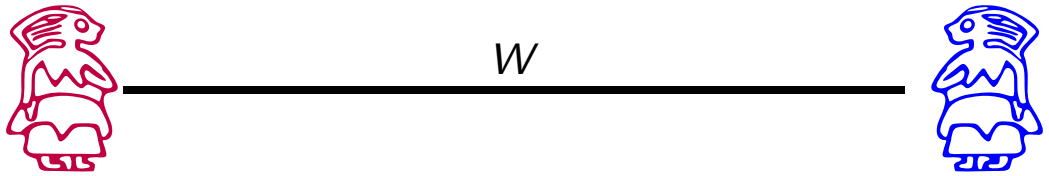
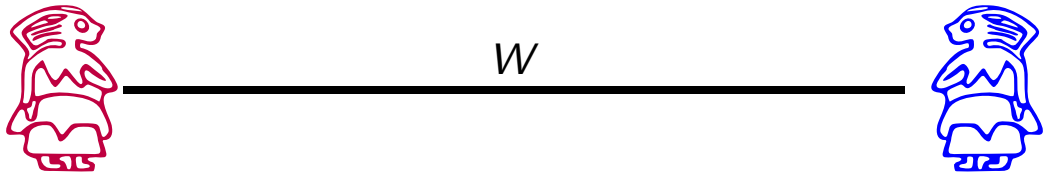


A



$$A \xrightarrow{W} B$$



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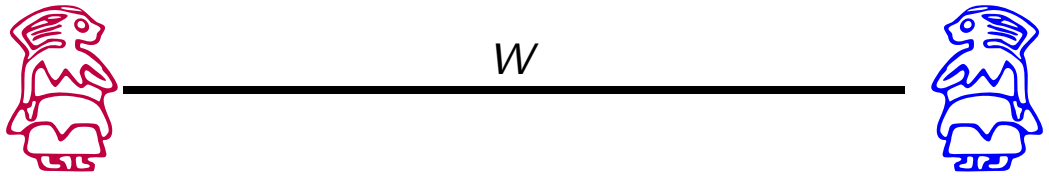

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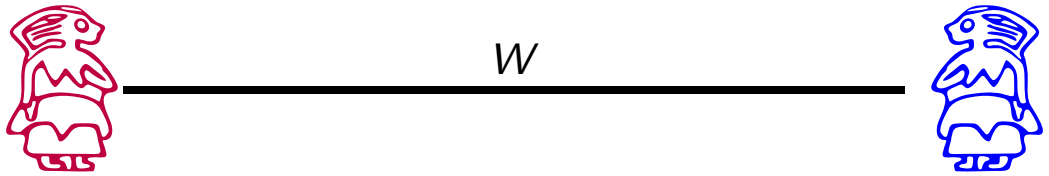

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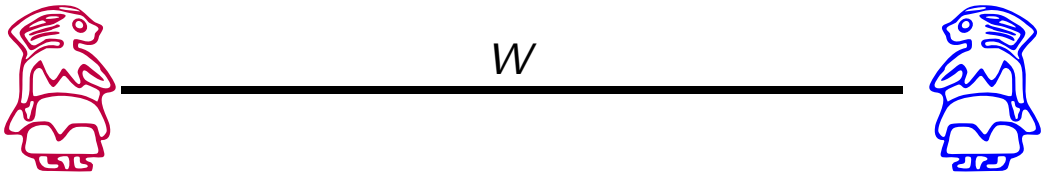

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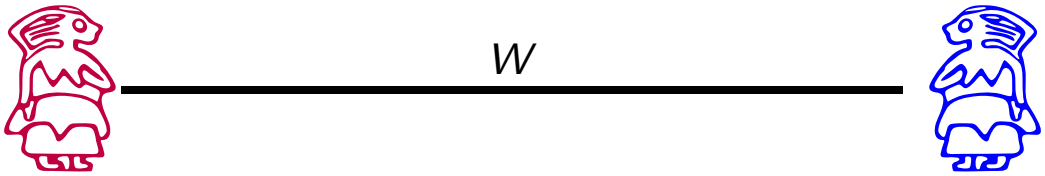

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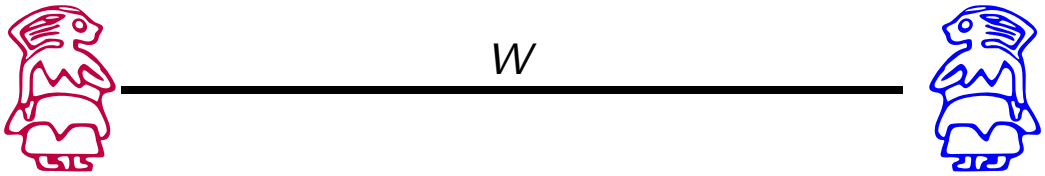

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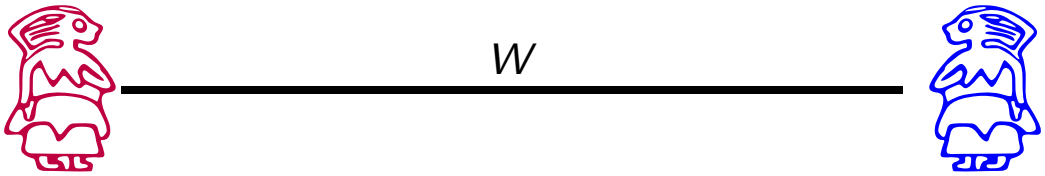

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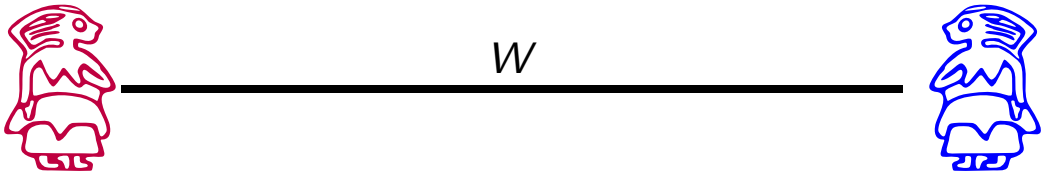
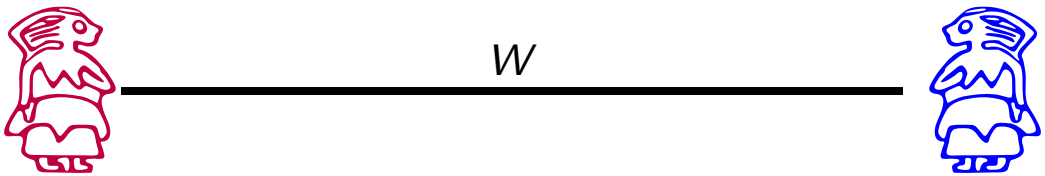
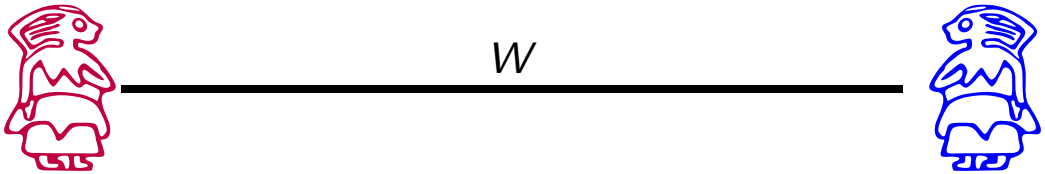


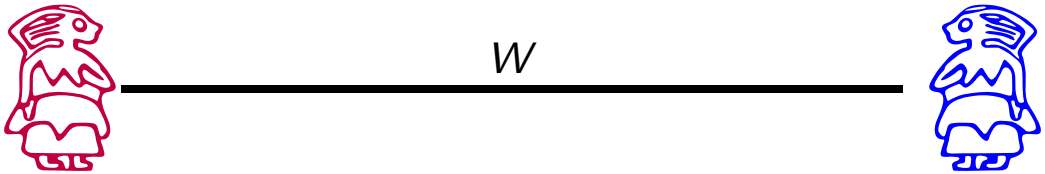
$$0 \xrightarrow{C} 0000$$



$$0 \xrightarrow{C} 0000 \xrightarrow{W} 0100$$



$$0 \xrightarrow{C} 0000 \xrightarrow{W} 0100 \xrightarrow[\substack{0000\\1111}]{}$$



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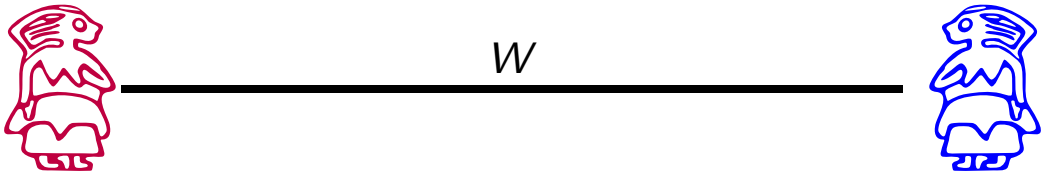

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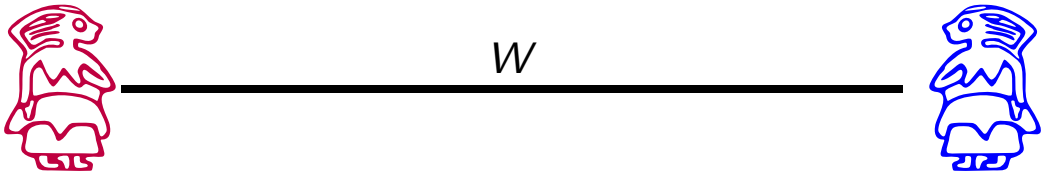

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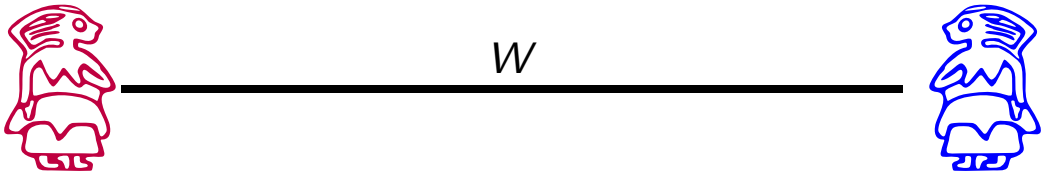

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$$P_e(C, W, D) = P(m \neq r)$$

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$$n \equiv \text{longitud}$$

$$k := \dim C$$

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$$\delta(C) = \min\{d(c, c') \mid c, c' \in C, c \neq c'\} = \min\{w(c) \mid c \in C \setminus \{0\}\}$$

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$$0000 \to 1111 \xrightarrow{D} 1 \ ☺$$

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	0	1
0	2/3	1/3
1	1/3	2/3

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Prop.

Supongamos $W(x|x) = p$ y $W(y|x) = \dfrac{1-p}{q-1}$. Si $p > \dfrac{1}{q}$, entonces MDD coincide con MLD.

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$$P(y_1^n|x_1^n) = \prod_{i=1}^{n} W(y_i|x_i) = p^{n-d(x,y)} \left(\frac{1-p}{q-1}\right)^{d(x,y)}$$

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Teorema de canal ruidoso [1948]

Dado cualquier canal W, para $\epsilon > 0$ y para $k < nI(W)$, si n es suficientemente grande, existe un código C de longitud n, $\dim C \geq k$ y un algoritmo de decodificación tal que la probabilidad de error de bloque es $\leq \epsilon$.

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- Turbo codes
- LDPC
- Concatenation codes.

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$$(u_1, u_2) \rightarrow u_1^2 G_A = (u_1 + u_2, u_2) \xrightarrow{W} (y_1, y_2)$$

$$W_1(y_1 y_2 | u_1 u_2) = W(y_1 | u_1 + u_2) W(y_2 | u_2)$$

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$$(10) \rightarrow (10) G_A = (1 + 0, 0) \xrightarrow{W} (01)$$

$$W_1(01|10) = W(0|1) W(1|0) = \frac{1}{9}$$

$$u_1 \rightarrow (u_1, u_2)G_A \xrightarrow{W} (y_1, y_2)$$

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$$W_1^{(2)}(y_1, y_2, u_1|u_2)$$

$$W_1$$

$$(u_1, u_2) \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \xrightarrow{W} (y_1, y_2)$$

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$$u_2 \xrightarrow{W_1^{(2)}} (y_1, y_2, u_1) \downarrow y_2$$

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$$P(y_2|u_2)$$

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$$P(y_2|u_2) = W(y_2|u_2)$$

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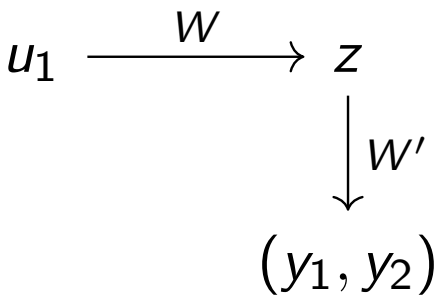
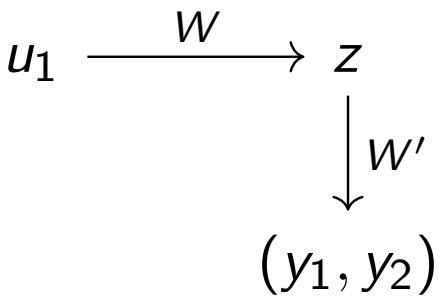
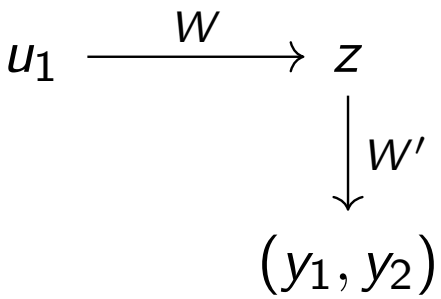
$$W \preceq W_1^{(2)}$$

$$I(W) \leq I(W_1^{(2)})$$


u_1
W
z
W'
(y_1, y_2)



$$P(y_1, y_2|u_1) = W_1^{(1)}(y_1, y_2|u_1)$$



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$$W_1^{(1)} \preceq W \preceq W_1^{(2)}$$

$$I(W_1^{(1)}) \leq I(W) \leq I(W_1^{(2)})$$

$$(u_1, u_2) \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \xrightarrow{W} (y_1, y_2)$$

$$A \in \mathcal{M}_{m\times n},\ B \in \mathcal{M}_{m'\times n'}$$

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{bmatrix}$$

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$$\text{¿} \lim_{n \to \infty} I\left(W_n^{(i_n)}\right)?$$

Polarización

Teorema [Arikan, 2008]

Sea $1 > \delta > 0$. Entonces

$$\lim_{n\to\infty} \frac{|\{i \in [n] \mid I(W_n^{(i)}) \in (1-\delta, 1]\}|}{2^n} = I(W)$$

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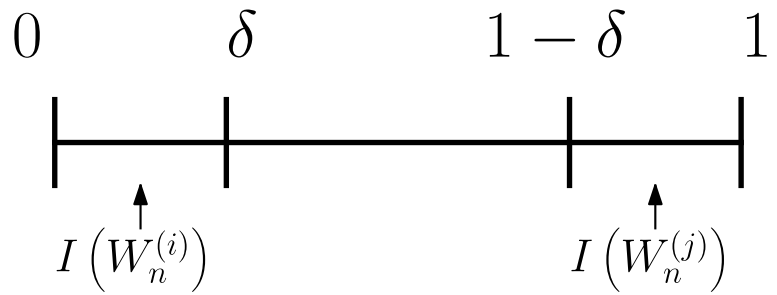

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$$C_{\{2,4\}} = \langle (1100), (1111) \rangle$$

Cancelación sucesiva

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$$O(n \log n)$$

$$\lim_{n\to\infty} P_e = 0$$

$$I(W_n^{(i)}) \geq \log_2 \frac{2}{1+2^{-n/2}},\ i \in \mathcal{A}_n$$

Teorema [Anderson, Matthews, 2014]

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