Seminario Virtual GASIULL

SEMIGRUPOS COHEN-MACAULAY Y SUS APLICACIONES

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Toric Mathematics from semigroup viewpoint

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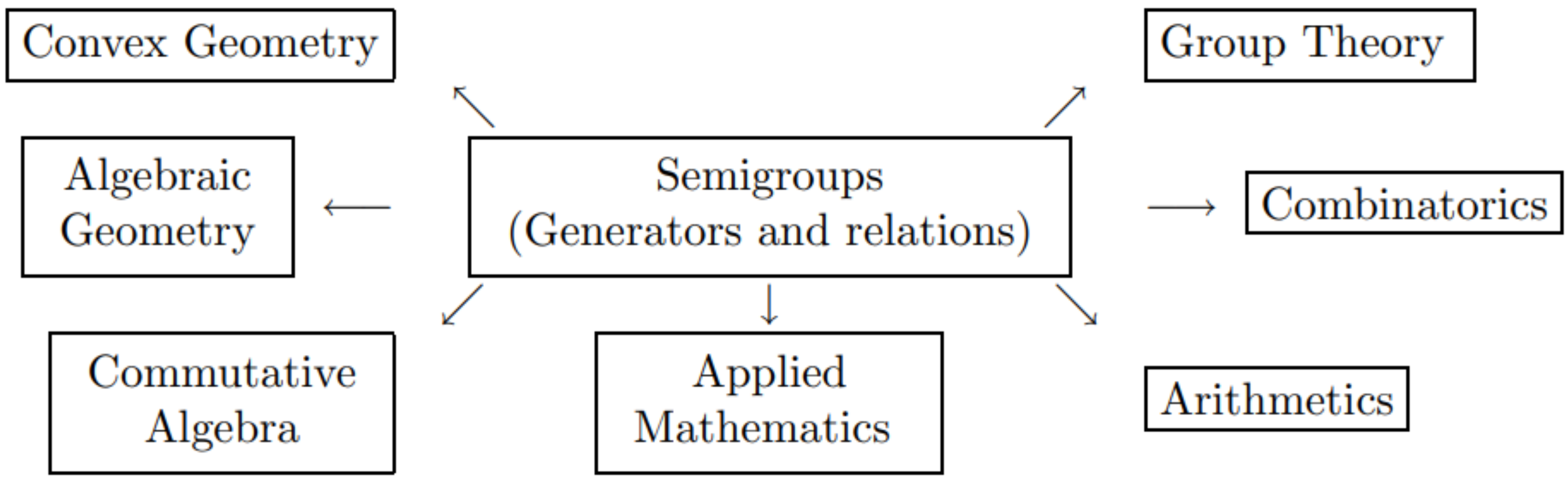

$$\pi_0 : \mathbb{N}^h \to S,$$

Semigroups

Let $\pi_0 : \mathbb{N}^h \to S$ be a surjective semigroup homomorphism where S is a cancellative commutative semigroup. Then, the following conditions are equivalent:

1. *$\pi_0^{-1}(m)$ is finite for every m.*
2. *There is no infinite sequence $m \in S$, $m_1, \ldots, m_i, \ldots \in S - \{0\}$, such that $m - m_1 - \cdots - m_i \in S$ for every i.*
3. *$S \cap (-S) = \{0\}$.*
4. *There exists a semigroup homomorphism $\lambda : S \to \mathbb{N}$ such that $\lambda(m) = 0$ iff $m = 0$.*

Groups

$$\pi_0 \; : \; \mathbf{N}^h \; \longrightarrow \; S$$

$$0 \rightarrow L \rightarrow G(\mathbf{N}^h) = \mathbf{Z}^h \rightarrow G(S) \rightarrow 0$$

$$0 \rightarrow L \rightarrow \mathbf{Z}^h \rightarrow \mathbf{Z}^h/L \rightarrow 0$$

Combinatorics

the congruence Γ and the lattice L

$(u, v) \in \Gamma$ then there is a unique element $w \in N^h$ such that $(u - w, v - w)$

$$b : \Gamma \to N^h \times L, \qquad b(u, v) = (w, u - v).$$

Algebra

$$\pi_0 : \mathbf{N}^h \rightarrow S$$

$$0 \rightarrow I \rightarrow A = k[\mathbf{N}^h] \stackrel{\varphi_0}{\rightarrow} R = k[S] \rightarrow 0$$

$$0 \longrightarrow F_p \stackrel{\varphi_p}{\longrightarrow} \cdots \longrightarrow F_2 \stackrel{\varphi_2}{\longrightarrow} F_1 \stackrel{\varphi_1}{\longrightarrow} F_0 = A \stackrel{\varphi_0}{\longrightarrow} R \longrightarrow 0,$$

$$V_i(m) = (N_i)_m/(M_A N_i)_m \qquad N_i = ker(\varphi_i)$$

$$V_i(m) = \tilde{H}_i(\Delta_m)$$

Convex

$$V_{\mathbf{Q}} := G(S) \otimes_{\mathbf{Z}} \mathbf{Q}$$

$C(S)$ $C(S) \cap -C(S) = 0)$

$$\Lambda = \mathcal{E} \cup \mathcal{C},$$

$\mathcal{T}_m$ J $\mathcal{E}$ $m - n_J \in S$

Δ_m J Λ $m - n_J \in S$

Arithmetics

Apery set relative to $\mathcal{E}$

set Q of elements of $q \in S$ such that $q - n \notin S$ for every $n \in \mathcal{E}$

$$\ldots \to H_{i+1}(Q_m) \to K_i \to \tilde{H}_i(\Delta_m) \to H_i(Q_m) \to K_{i-1} \to \ldots$$

Geometry

Let S be a finitely generated cancellative commutative semigroup. Assume that S is endowed with a semigroup map $\lambda : S \to \mathbf{N}$ such that the semigroup is generated by the elements in the set $S_1 = \lambda^{-1}(1)$. Then, for any choice of the field k, the couple (S, λ) gives rise to an (abstract) $(d-1)$-dimensional projective algebraic scheme, namely $Z = Proj(k[S])$, where now $k[S]$ is seen as an $\mathbf{N}$-graded algebra by relaxing its natural S-grading via the map λ (in other words, degree $i \in \mathbf{N}$ homogeneous elements are the sums of homogeneous elements of S-degrees in $\lambda^{-1}(i)$). Along the paper couples (S, λ) as above will be related as polarized semigroups.

Depth and Cohen Macaulay

r=depth d=dimensión e= edges

integers r,d and e associated to a positive semigroup are such that $r \leq d \leq e$. The integers d and e are easily recognized from S. To recognize r, in [8] it is proved that if r_0 is a integer with $1 \leq r_0 \leq d$, then one has that the inequality $r \geq r_0$ is equivalent to the fact $\tilde{H}_{e-r_0}(\mathcal{T}_m) = 0$ for every $m \in S$. In particular, for $r_0 = d$ one gets a characterization of Cohen-Macaulayness by the property

$$\tilde{H}_{e-d}(\mathcal{T}_m) = 0$$

for every $m \in S$, which, for the simplicial case $e = d$ means that all the complexes $\mathcal{T}_m$ are connected. From this, it is easy to recover the well known characterization due to Goto [12] with asserts that, for simplicial semigroups, Cohen-Macaulay is equivalent to the property

$$m \in G(S), n, n' \in \mathcal{E}, n \neq n', m + n \in S, m + n' \in S \Rightarrow m \in S.$$

ffine versus Projective

An affine toric curve is given by the numerical semigroup S given by a set of h nonnegative integers. One has $r = d = 1$ and, since the cone $C(S)$ has ly one edge, also $e = 1$. Thus, this case is a simplicial Cohen-Macaulay one.

$$\tilde{H}_i(\Delta_m) = H_i(Q_m)$$

A projective toric curve of degree s is given by a subsemigroup of $\mathbf{N}^2$ gen- ted by a set $\Lambda = \mathcal{E} \cup \mathcal{C}$, where $\mathcal{E}$ is the set consisting of the two elements 0) and $(0, s)$ and $\mathcal{C}$ consists of elements $(c_1, s - c_1), \ldots, (c_{h-2}, s - c_{h-2})$ for ferent values c_i with $0 < c_i < s$. The semigroup S can be polarized by the ction λ given by $\lambda(c, c') = (c + c')/s$, so that S defines an embedding of the ojective toric curve in $\mathbf{P}^{h-1}$. Notice that one has $d = e = 2$ and that either $= 2$ or $r = 1$, according as the projective curve be or not be arithmetically hen-Macaulay.

$$\ldots \to H_{i+1}(Q_m) \to H_i(D_m) \to \tilde{H}_i(\Delta_m) \to H_i(Q_m) \to \ldots.$$

Applied

Assume one has a coin system with coins of values
$< c_2 < \ldots < c_{h-1}$. Then, setting $s = c_{h-1}$ one has a projective toric curve Z,
mely that of degree s given by the (polarized) subsemigroup of $\mathbb{N}^2$ generated
the elements $(0, s), (c_1, s - c_1), \ldots, (c_{h-1}, s - c_{h-1}) = (s, 0)$.

. If the greedy algoritm works, then the polarized semigroup is CM.
2. The greed algorithm works if and only iff the polarized semigroups of all
nitial segments of coins are CM.

Let S_1 be the numerical semigroup generated by $c_1, \ldots, c_{h-2}, s$, and for each $c \in S_1$ denote by $\mu(c)$ the least number of the generators the above generators of S_1 needed the achieve the sum c. Notice, that the function μ satisfies the property $\mu(c) \leq \mu(c-s)+1$ for every $c \in S$ whenever $c - s \in S$. By translating into arithmetics the methods in [8], in [9] it is shown that the projective toric curve is arithmetically Cohen-Macaulay if and only if one has $\mu(c) = \mu(c-s)+1$ for every $s \in S_1$ such that $c - s \in S$.

Computing

Theorem *Let* $\mathcal{S} = \langle \mathcal{A} \rangle = \langle \mathbf{a}_1, \ldots, \mathbf{a}_n \rangle \subseteq \mathbb{Z}^m \oplus T$ *be a reduced monoid,* $B = \{\mathbf{b}_1, \ldots, \mathbf{b}_s\} \subseteq \mathcal{S} \setminus \{\mathbf{0}\}$ *and* $I = \cup_{i=1}^{s}(\mathbf{b}_i + \mathcal{S})$*. The following statements are equivalent:*

(1) *The Apéry set* $\mathrm{Ap}_{\mathcal{S}}(B)$ *is finite.*

(2) $\mathcal{C}_{\mathcal{A}} = \mathcal{C}_B$.

(3) $I \cup \{\mathbf{0}\}$ *is a (finitely generated) reduced monoid.*

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