Introduction

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Newton polygon of the discriminant

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Plan of the talk

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1. Newton polygon and initial Newton polynomial
2. Direct image of the germ of a plane analytic curve
3. Jacobian curve and discriminant curve
4. Main theorem
   - Key ingredients of the proof
   - Corollaries
5. Bibliography

Basic notions

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Any germ $f : (\mathbb{C}^2, 0) \to \mathbb{C}$ of a holomorphic function can be identified with the power series

$$f = \sum a_{ij} x^i y^j \in \mathbb{C}[[x, y]]$$

convergent in a neighborhood of zero.

The *Newton polygon* $\Delta(f)$ is by definition the convex hull of the union

$$\bigcup_{a_{ij} \neq 0} \left((i, j) + \mathbb{R}^2_{\geq 0}\right).$$

If S is the union of all compact edges of $\Delta(f)$, then the polynomial

$$f|_S := \sum_{(i,j) \in S} a_{ij} x^i y^j$$

is called the *initial Newton polynomial* of f.

Example

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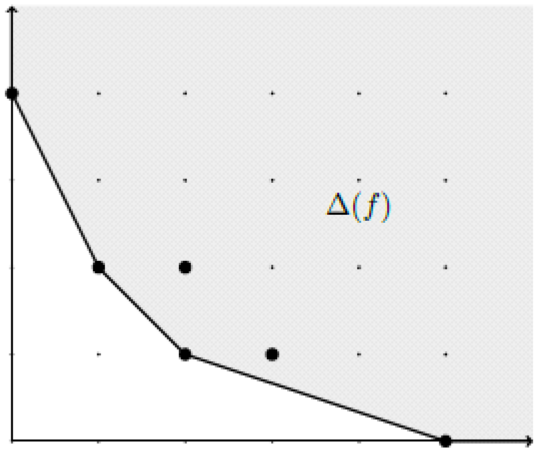

$$f = y^4 + 2xy^3 - x^2y^3 + x^2y + 5x^3y + x^5$$

$NP(f) = y^{4} + 2x$

Factorization

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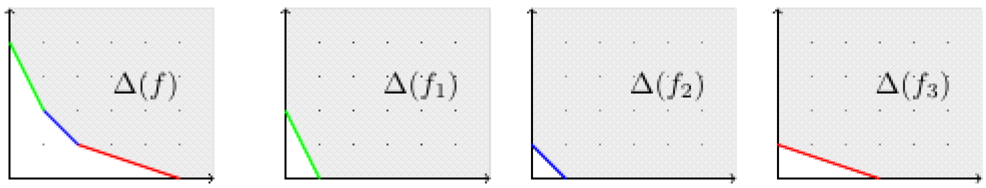
If $f \in \mathbb{C}[[x, y]]$ then there exist a factorization $f = f_1 \cdots f_k$ such that the Newton polygon of each factor is *elementary*. The Newton polygons of the factors are in one-to-one correspondence with the compact edges of $\Delta(f)$.

Example



$f = f_1 \cdot f_2 \cdot f_3$

Factorization

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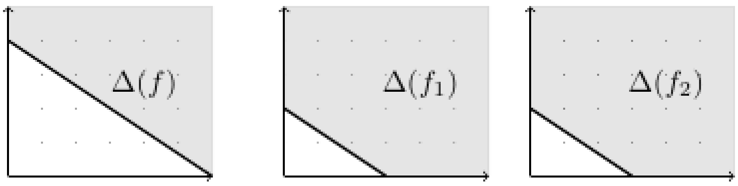
If $f \in \mathbb{C}[[x, y]]$ has an elementary Newton polygon and

$$\mathrm{NP}(f) = \prod_{i=1}^{n} (y^k - t_i x^l)^{\nu_i},$$

where $t_i \neq 0$ and $t_i \neq t_j$ for $i \neq j$, then there exists a factorization $f = f_1 \cdots f_n$ such that $\mathrm{NP}(f_i) = (y^k - t_i x^l)^{\nu_i}$ for $i = 1, \ldots n$.

Example



$\mathrm{NP}(f) = y^2 - x^3)(y^2 + 2x^3)$ $\quad f = f_1 \cdot f_2$

$\mathrm{NP}(f_1) = y^2 - x^3, \quad \mathrm{NP}(f_2) = y^2 + 2x^3$

Factorization

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The initial Newton polynomial of f allows to find a factorization $f = f_1 \cdots f_n$ such that the Newton polygon of each factor is elementary and the initial Newton polynomial of each factor is a power of an irreducible polynomial.

Direct image

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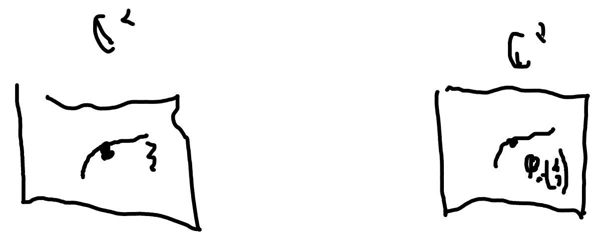
Let $\phi = (f, g)\colon (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ be the germ of a holomorphic mapping with an isolated zero. To any germ ξ of an analytic curve in $(\mathbb{C}^2, 0)$ one associates its *direct image* $\phi_*(\xi)$. The direct image of ξ by ϕ is an analytic curve germ in the target space uniquely determined by the following two properties:

(i) if $\xi \subset (\mathbb{C}^2, 0)$ is an irreducible curve then $\phi_*(\xi)$ is the curve of equation $H^d = 0$, where $H = 0$ is a reduced equation of the curve $\phi(\xi)$ in the target space and d is the topological degree of the restriction $\phi|_\xi : \xi \to \phi(\xi)$.

(ii) if $h = h_1 \cdots h_s$ is a factorization of a power series h to the product of irreducible factors in $\mathbb{C}\{x, y\}$, then $\phi_*(\{h = 0\})$ is the curve $H_1 \cdots H_s = 0$, where the curves $H_i = 0$ are the direct images of the branches $h_i = 0$ for $i = 1, \ldots, s$.



Discriminant curve

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Let $\phi = (f, g)\colon (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ be the germ of a holomorphic mapping with an isolated zero. We call

$$\mathrm{Jac}(\phi) = \frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x} = 0$$

the *Jacobian curve* of ϕ and the direct image of the Jacobian curve is called the *discriminant curve.* The Newton diagram of the discriminant, denoted $Q(f, g)$, is called the *Jacobian Newton diagram* of (f, g). The notion of a Jacobian Newton polygon was introduced by Tessier.

Factorization

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Let $J = 0$ be the Jacobian curve and $D = 0$ be the discriminant curve of $\phi = (f, g)\colon (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$. Any factorization of the discriminant

$$D = D_1 \cdots D_n$$

induces a factorization of the Jacobian

$$\mathrm{Jac}(\phi) = J_1 \cdots J_n$$

such that $\phi_*(\{J_i = 0\}) = \{D_i = 0\}$ for $i = 1, \ldots, n$.

Polar quotients

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If D_i has an elementary Newton polygon then its compact edge intersects coordinate axes at points $(i_0(g, J_i), 0)$ and $(0, i_0(f, J_i))$. Moreover for any irreducible factor p of J_i we have

$$\frac{i_0(g,p)}{i_0(f,p)} = \frac{i_0(g,J_i)}{i_0(f,J_i)}$$

The Jacobian curve in the case of $f = 0$ smooth and transverse to $g = 0$ is called the generic polar curve of g. In this case the quotients $i_0(g,p)/i_0(f,p)$ where h is an irreducible factor of $\mathrm{Jac}(\phi)$, are called the *polar quotients*.

Example

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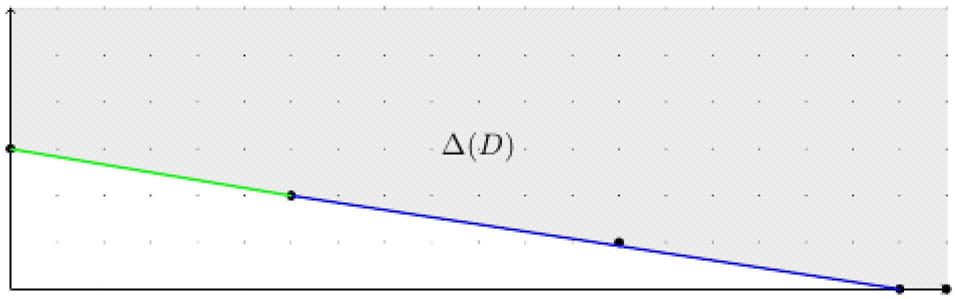
Let $f = x$ and $g(x,y) = (y^2 - x^3)^2 - x^5 y$. Then the discriminant of $\phi = (f,g)$ is equal to

$$D(u,v) = \mathrm{discr}_y(g(u,y) - v) = -256v^3 + 256u^6v^2 + 288u^{13}v - 256u^{19} - 27u^{20}.$$



$D = D_1 \cdot D_2$, $\mathrm{Jac}(\varphi) = J_1 \cdot J_2$

$i_0(J_1, x) = 1$, $i(J_1, y) = 6$

$i_0(J_1, x) =$ $i_0(J_2, y) =$

Polar case

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Let $\phi = (f, g)\colon (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ be the germ of a holomorphic mapping such that the curve $f = 0$ is smooth and transverse to $g = 0$. Then:

Polar case

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Let $\phi = (f, g)\colon (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ be the germ of a holomorphic mapping such that the curve $f = 0$ is smooth and transverse to $g = 0$. Then:

- [Kuo, Lu] The polar quotients are invariants of singularity of the curve $g = 0$.
- [Merle] If g is irreducible then the Jacobian Newton polygon $Q(f, g)$ is an invariant of singularity of the curve $g = 0$. Moreover $Q(f, g)$ determines the equisingularity type of the curve $g = 0$.
- [Teissier], [Eggers], [Kuo, Lu] The Jacobian Newton polygon $Q(f, g)$ is an invariant of singularity of the curve $g = 0$.
- [JG, EGB] If $Q(f, g) = Q(\tilde{f}, \tilde{g})$ and g is irreducible then $\tilde{g}$ is irreducible. Moreover the curves $g = 0$ and $\tilde{g} = 0$ are equisingular.

Equisingularity

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The pairs of curves $f = 0$, $g = 0$ and $\tilde{f} = 0$, $\tilde{g} = 0$ are *equisingular* if there exist factorizations $f = h_1 \cdots h_s$, $g = h_{s+1} \cdots h_r$, $\tilde{f} = \tilde{h}_1 \cdots \tilde{h}_{\tilde{s}}$, $\tilde{g} = \tilde{h}_{\tilde{s}+1} \cdots \tilde{h}_{\tilde{r}}$ into the product of irreducible factors in $\mathbb{C}\{x, y\}$ such that

- $s = \tilde{s}$, $r = \tilde{r}$,
- for $i = 1, \ldots, r$, the semigroups $\Gamma(h_i) := \{i_0(h_i, w) : w \notin (h_i)\}$ and $\Gamma(\tilde{h}_i) := \{i_0(\tilde{h}_i, w) : w \notin (\tilde{h}_i)\}$ are equal,
- $i_0(h_i, h_j) = i_0(\tilde{h}_i, \tilde{h}_j)$ for $1 \leq i < j \leq r$.

Invariance of the Jacobian Newton polygon

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Theorem 1 ([Michel], [JG]) *Let* $(f,g)\colon (\mathbb{C}^2,0) \longrightarrow (\mathbb{C}^2,0)$ *be a holomorphic mapping with an isolated zero. Then the Jacobian Newton polygon* $Q(f,g)$ *depends only on the equisingularity type of the pair of curves* $f=0$, $g=0$.

Main theorem

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Theorem 2 ([Gryszka, JG, Parusiński]) *Let* $f, g, u', u'' \in \mathbb{C}\{x, y\}$ *be convergent power series vanishing at zero such that* f *and* g *are coprime and let* $\tilde{f} = (1 + u')f$, $\tilde{g} = (1 + u'')g$. *Then the initial Newton polynomials of discriminants of mappings* $(f, g) : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ *and* $(\tilde{f}, \tilde{g}) : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ *are equal.*

Ingredients of the proof

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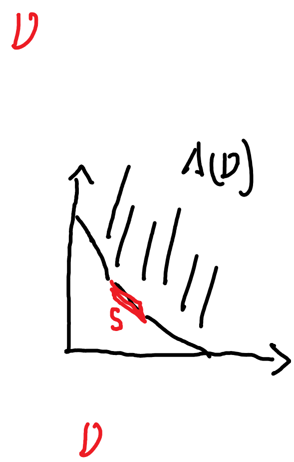

Let $D(u, v)$ be a nonzero complex power series and let $w = (k, l)$ be a weight vector, where k, l are coprime positive integers. Then D can be written as the sum of quasi-homogeneous polynomials $D = D_m + D_{m+1} + \cdots$, where $D_m \neq 0$ and $\deg_w D_i = i$ for $i \geq m$. Write D_m as a product

$$D_m(u, v) = C u^{\nu_0} v^{\nu_{n+1}} \prod_{i=1}^{n} (v^k - t_i u^l)^{\nu_i}, \tag{1}$$

where $t_i \neq 0$ and $t_i \neq t_j$ for $i \neq j$.

Lemma 3 *Let $H_t = (v^k - tu^l)^N - u^{l(N+1)}$. Then for a sufficiently large integer N and for every $t \in \mathbb{C}^*$ such that $t \neq t_i$ for $1 \leq i \leq n$, one has*

$$\nu_j kl = i_0(D, H_{t_j}) - i_0(D, H_t).$$

Casas-Alvero formula

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Theorem 4 (Casas-Alvero) *Let* $(f, g) : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ *be the germ of a holomorphic mapping with an isolated zero. Let* $D(u, v) = 0$ *be the discriminant of* (f, g)*. Take any curve germ* $H(u, v) = 0$ *and let* $h(x, y) = H(f(x, y), g(x, y))$*. Then*

$$\mu(h) - 1 = i_0(f, g)[\mu(H) - 1] + i_0(D, H),$$

where $\mu(h)$ *denotes the Milnor number of the curve* $h = 0$ *at zero.*

Corollary

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Corollary 5 *Let $(f, g) : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ be the germ of a holomorphic mapping with an isolated zero. Let $D(u, v) = 0$ be the discriminant curve of (f, g) and $h_t = (g^k - tf^l)^N - f^{l(N+1)}$ for $N > 1$. Then, under the notation of (1), for $N \gg 1$ and $t \in \mathbb{C}^*$ different from $t_1, \ldots, t_n$, we have $\nu_j kl = \mu(h_{t_j}) - \mu(h_t)$.*

$$H_t = (v^k - tu^l)^N - u^{l(N+1)}, \qquad h_t = (g^k - tf^l)^N - u^{l(N+1)}$$

$$\mu(h_t) - 1 = i_0(f, g)[\mu(H_t) - 1] + i_0(D, H_t)$$

$$\mu(h_{t_i}) - 1 = i_0(f, g)[\mu(H_{t_i}) - 1] + i_0(D, H_{t_i})$$

$$\mu(h_{t_i}) - \mu(h_t) = i_0(D, H_{t_i}) - i(D, H_t) = \nu_i k l$$

$$\mu(\tilde{h}_{t_i}) - \mu(\tilde{h}_t) = \mu_i k l$$

Key lemma

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Lemma 6 (Key Lemma) *Let $f, g, u', u'' \in \mathbb{C}\{x, y\}$ be convergent power series vanishing at zero such that f and g are coprime and let $\tilde{f} = (1 + u')f$, $\tilde{g} = (1 + u'')g$. Then for sufficiently large integer N the curves $(g - f)^N - f^{N+1} = 0$ and $(\tilde{g} - \tilde{f})^N - \tilde{f}^{N+1} = 0$ are equisingular.*

Corollary 7 *Let $f, g, u', u'' \in \mathbb{C}\{x, y\}$ be convergent power series vanishing at zero such that f and g are coprime and let $\tilde{f} = (1 + u')f$, $\tilde{g} = (1 + u'')g$. Then for positive integers k, l, $t \neq 0$ and sufficiently large integer N the curves $h_t = (g^k - tf^l)^N - f^{l(N+1)} = 0$ and $\tilde{h}_t = (\tilde{g}^k - t\tilde{f}^l)^N - \tilde{f}^{l(N+1)} = 0$ are equisingular.*

$g_1 = g^k$, $f_1 = tf^l$

Proof of the main theorem

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Let $D(u,v)=0$ be the discriminant of $(f,g):(\mathbb{C}^2,0)\to(\mathbb{C}^2,0)$ and let $\tilde{D}(u,v)=0$ be the discriminant of $(\tilde{f},\tilde{g}):(\mathbb{C}^2,0)\to(\mathbb{C}^2,0)$. Let $w=(k,l)$ be an arbitrary weight vector, where k, l are coprime positive integers. Write $\mathrm{in}_w D = Cu^{\nu_0}v^{\nu_{n+1}}\prod_{i=1}^{n}(v^k-t_iu^l)^{\nu_i}$ and $\mathrm{in}_w\tilde{D}=\tilde{C}u^{\eta_0}v^{\eta_{n+1}}\prod_{i=1}^{n}(v^k-t_iu^l)^{\eta_i}$. It is enough to prove that $\nu_i=\eta_i$ for $1\le i\le n$. This follows from Corollary 5 since by Corollary 7 for $t\neq 0$ one has $\mu(h_t)=\mu(\tilde{h}_t)$. This ends the proof.

Corollaries

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Corollary 8 *Let* $(f,g)\colon (\mathbb{C}^2,0) \longrightarrow (\mathbb{C}^2,0)$ *be a holomorphic mapping with an isolated zero. Then the initial Newton polynomial of its discriminant is determined, up to rescalling variables, by the ideals* (f) *and* (g).

Corollaries

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Corollary 9 *Under the assumptions of Theorem 2, for every pair of coprime positive integers k, l we have*

(i) *the pencils $g^k - tf^l = 0$ and $\tilde{g}^k - t\tilde{f}^l = 0$, where $t \in \mathbb{C}$ is a parameter, have the same sets of atypical values.*

(ii) *the meromorphic functions g^k/f^l and $\tilde{g}^k/\tilde{f}^l$ have the same asymptotic critical values.*

(iii) *the generic fibers of g^k/f^l and $\tilde{g}^k/\tilde{f}^l$ are equisingular.*

Bibliography

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[Ca1] E. Casas-Alvero, *Discriminant of a morphism and inverse images of plane curve singularities*, Math. Proc. Cambridge Philos. Soc. **135** (2003), no. 3, 385–394.

[Ca2] E. Casas-Alvero, *Local geometry of planar analytic morphisms*, Asian J. Math. **11** (2007), no. 3, 373–426.

[Eg] H. Eggers, *Polarinvarianten und die Topologie von Kurvensingularitäten*, Bonner Math. Schriften 147, Universität Bonn, Bonn 1982.

[GB-G] E. García Barroso and J. Gwoździewicz. *A discriminant criterion of irreducibility*. Kodai Mathematical Journal, **35(2)** (2012), 403-414.

[Gw] J. Gwoździewicz, *Invariance of the Jacobian Newton diagram*, Math. Res. Lett. **19** (2012), no. 2, 377–382.

[KL] T-C. Kuo, Y.C. Lu, *On analytic function germ of two complex variables*, Topology 16 (1977), 299–310.

[KP] T.-C. Kuo, A. Parusiński, *Newton Polygon relative to an arc*, in Real and complex singularities (São Carlos, 1998), vol. **412** of Chapman & Hall/CRC Res. Notes Math., Chapman & Hall/CRC, 2000, pp. 76–93.

Bibliography

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[Ma] H. Maugendre, *Discriminant d'un germe* $(g, f) : (C^2, 0) \to (C^2, 0)$ *et quotients de contact dans la résolution de* $f \cdot g$, Ann. Fac. Sci. Toulouse Math. (6) 7 (3) (1998), 497–525.

[Me] M. Merle, *Invariants polaires des courbes planes*, Invent. Math. 41 (1977), 103–111.

[Mi] F. Michel, *Jacobian curves for normal complex surfaces*, Contemporary Mathematics, Volume 475 (2008), 135–150.

[Te1] B. Teissier, *The hunting of invariants in the geometry of discriminants. Real and complex singularities* Proc. Ninth Nordic Summer School/NAVF Sympos. Math., Oslo, 1976, pp. 565–678.

[Te2] B. Teissier, *Polyèdre de Newton jacobien et équisingularité, Séminaire sur les Singularités*, Publ. Math. Univ. Paris VII, **7** (1980), 193–221.

[Za1] O. Zariski, *Studies in equisingularity. I. Equivalent singularities of plane algebroid curves*, Amer. J. Math. **87** (1965), pp. 507–536.

[Za2] O. Zariski, *Studies in equisingularity. II. Equisingularity in codimension 1 (and characteristic zero)*, Amer. J. Math. **87** (1965), pp. 972–1006.