Analytic invariants of a plane branch and their semiroots

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Analytic Plane Curve

Given $f \in \langle X, Y\rangle \subset \mathbb{C}\{X, Y\}$ the **curve** defined by f is (the germ of set)

$$\mathcal{C}_f = \{(a, b) \in \mathbb{C}^2;\ f(a, b) = 0\}.$$

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If f is irreducible then we say that $\mathcal{C}_f$ is **irreducible**.

We say that $\mathcal{C}_f$ is singular (at $(0, 0) \in \mathbb{C}^2$) if

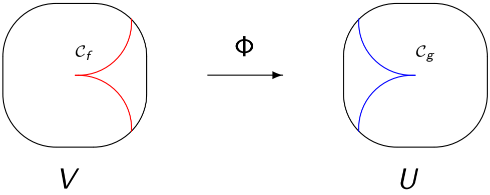
$$f_x(0, 0) = f_y(0, 0) = 0.$$

In what follows we will only consider singular irreducible plane curve that we call a **plane branch**.

Topological and Analytic Equivalence

Topological Equivalence:

$$\mathcal{C}_f \equiv_{Top} \mathcal{C}_g \quad \Leftrightarrow \quad \Phi(V \cap \mathcal{C}_f) = U \cap \mathcal{C}_g$$



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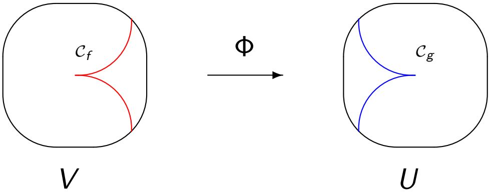

Analytic Equivalence:
If Φ is an *analytic isomorphism* then $\mathcal{C}_f \equiv_{Ana} \mathcal{C}_g$.

Local Ring

$$\mathcal{C}_f \equiv_{Ana} \mathcal{C}_g \quad \Leftrightarrow \quad f \overset{\mathcal{K}}{\sim} g \quad \Leftrightarrow \quad \Psi(f) = u.g$$

$\Psi \in Aut(\mathbb{C}\{X,Y\})$ and $u \in \mathbb{C}\{X,Y\}$ a unit.

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$$\mathcal{C}_f \equiv_{Ana} \mathcal{C}_g \quad \Leftrightarrow \quad \mathcal{O}_f \cong \mathcal{O}_g. \quad (\text{as } \mathbb{C} - \text{algebras})$$

Parametrizations

If $mult(f) = n$, i.e. $f \in \langle X, Y \rangle^n \setminus \langle X, Y \rangle^{n+1}$, then by the WPT,

$$u.\Psi(f) = Y^n + a_1(X)Y^{n-1} + \ldots + a_n(X) \in \mathbb{C}\{X\}[Y].$$

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Newton-Puiseux parametrization:

$$\varphi_f(t) = (x(t), y(t)) \in \mathbb{C}\{t\} \times \mathbb{C}\{t\}$$

We have the exact sequence

$$\begin{array}{ccccccccc} \{0\} & \rightarrow & \langle f \rangle & \rightarrow & \mathbb{C}\{X, Y\} & \rightarrow & \mathbb{C}\{x(t), y(t)\} & \rightarrow & \{0\} \\ & & & & h & \mapsto & h(x(t), y(t)) & & \end{array}$$

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$$\varphi_f \overset{\mathcal{A}}{\sim} \varphi_g \quad \Leftrightarrow \quad \varphi_g = \sigma \circ \varphi_f \circ \rho^{-1} \quad \Leftrightarrow \quad \begin{array}{ccccc} & \mathbb{C} & \overset{\varphi_f}{\longrightarrow} & \mathbb{C}^2 & \\ \rho & \downarrow & \circlearrowright & \downarrow & \sigma \\ & \mathbb{C} & \overset{\varphi_g}{\longrightarrow} & \mathbb{C}^2 & \end{array}$$

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$$f_2 = Y^3 - X^7 + X^5 Y$$

$$\begin{gathered} f_3 = Y^8 + (1 - 23X)Y^7 + (-14X + 70X^2)Y^6 + (84X^2 - 196X^3)Y^5 + \\ +(-1 - 280X^3 + 280X^4)Y^4 + (-1 + 2X + 560X^4 - 112X^5)Y^3 + \\ +(3X - 672X^5 - 224X^6)Y^2 + (-3X^2 - 2X^3 + 448X^6 + 320X^7)Y + \\ +X^3 + X^4 - 128X^7 - 128X^8 \end{gathered}$$

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$$\mathcal{C}_{f_1} \not\equiv_{Ana} \mathcal{C}_{f_2} \not\equiv_{Ana} \mathcal{C}_{f_3} \qquad \mathcal{C}_{f_1} \equiv_{Ana} \mathcal{C}_{f_3}.$$

$f_3 = u.\Psi(f_1)$ with $u = 1+X+Y$ and $\Psi(X,Y) = (X+Y, X-Y)$.

Some invariants

Milnor Number

$$\mu_f = dim_{\mathbb{C}} \frac{\mathbb{C}\{X, Y\}}{\langle f_x, f_y \rangle}$$

$$\mathcal{C}_f \equiv_{Top} \mathcal{C}_g \quad \Rightarrow \quad \mu_f = \mu_g.$$

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Semigroup Γ_f

$$\Gamma_f = \left\{ \nu_f(h) := dim_{\mathbb{C}} \frac{\mathbb{C}\{X,Y\}}{\langle f, h \rangle} = ord_t h(\varphi_f);\ \ h \in \mathbb{C}\{X,Y\} \setminus \langle f \rangle \right\}$$

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Brauner, Burau, Milnor, Zariski

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Generators of Γ_f

Given $\varphi_f(t) = \left(t^n, \sum_{i \geq n} a_i t^i\right)$ we put $\beta_0 := e_0 := n$;

$$\beta_i = \min\{j; a_j \neq 0 \ and \ e_{i-1} \nmid j\} \quad e_i = gcd(e_{i-1}, \beta_i) \ \text{ for } \ i > 0.$$

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There exists $g \geq 1$ such that $e_g = 1$ and

$$\Gamma_f = \langle v_0, v_1, \ldots, v_g \rangle$$

with

$$v_0 := \beta_0, \quad v_1 := \beta_1, \quad v_{i+1} = \frac{e_{i-1}}{e_i} v_i + \beta_{i+1} - \beta_i \quad 1 \leq i < g.$$

Consider $\Omega^1 = \mathbb{C}\{X,Y\}dX + \mathbb{C}\{X,Y\}dY$, $\varphi_f = (x(t), y(t))$ and

$$\begin{array}{cccc} \varphi_f^*: & \Omega^1 & \longrightarrow & \mathbb{C}\{t\} \\ & \omega = gdX + hdY & \mapsto & t\cdot(g(\varphi_f)x(t)' + h(\varphi_f)y(t)'). \end{array}$$

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We have that

$$\ker \varphi_f^* = \{\omega \in \Omega^1;\ \varphi_f^*(\omega) = 0\} = \left\{\omega \in \Omega^1;\ \frac{\omega \wedge df}{dX \wedge dY} \in \langle f \rangle\right\} = f \cdot \Omega(log\ \mathcal{C})$$

where $\Omega(log\ \mathcal{C})$ is **logarithmic differential forms** along $\mathcal{C}$ which is the dual of the **logarithmic vector field** $Der(-log\ \mathcal{C})$ along $\mathcal{C}$.

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If $\mathcal{F} := f \cdot \Omega^1 + \mathbb{C}\{X,Y\} \cdot df$, then the **Kähler differential module** of $\mathcal{O}_f$ is

$$\Omega_f \approx \frac{\Omega^1}{\mathcal{F}}$$

and its torsion submodule is $\quad \mathcal{T}_f \approx \dfrac{f \cdot \Omega(\log \mathcal{C})}{\mathcal{F}}.$

Set Λ_f (a finer analytic invariant)

Consider $\Omega^1 = \mathbb{C}\{X,Y\}dX + \mathbb{C}\{X,Y\}dY$ and the map

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$\tau_f = \ell(\mathcal{T}_f) = \mu_f - \sharp \Lambda_f \setminus \Gamma_f$ (Pikham, Zariski, Berger)

Theorem (Hefez, —- . *Standard bases for local rings of branches and their modules of differentials*. J. Symb. Comp. 42, 2007)

We have an algorithm (with input φ_f or f) to compute Λ. It is possible to compute all set Λ for branches with a semigroup Γ fixed.

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Theorem (Hefez, —–. *The analytic classification of plane branches*. Bull. Lond. Math. Soc. 43(2), 2011)

$$\varphi_f \approx (t^{v_0}, t^{v_1}) \quad \textit{iff} \quad \Lambda_f \setminus \Gamma_f = \emptyset \quad \textit{or}$$

$$\varphi_f \approx \left(t^{v_0}, t^{v_1} + t^{\lambda} + \sum_{\substack{i>\lambda \\ i \notin \Lambda_f - v_0}} a_i t^i \right) \quad \textit{with} \quad \lambda + v_0 = \min \Lambda_f \setminus \Gamma_f .$$

Moreover,
$(t^{v_0}, t^{v_1} + t^{\lambda} + \sum_{\substack{i>\lambda \\ i \notin \Lambda_f - v_0}} a_i t^i) \approx (t^{v_0}, t^{v_1} + t^{\lambda} + \sum_{\substack{i>\lambda \\ i \notin \Lambda_f - v_0}} b_i t^i) \Leftrightarrow$
$\exists\ c \in \mathbb{C};\ \ c^{\lambda - v_1} = 1$ *such that* $a_i = c^{i-v_1} b_i$.

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A polynomial $f_k \in \mathbb{C}\{X\}[Y]$ for $0 \leq k < g$ is called a k-**semiroot** of f if f_k is monic,

$$deg(f_k) = \frac{v_0}{gcd(v_0, \dots, v_k)} = \frac{v_0}{e_k} \text{ and } \nu_f(f_k) = ord_t(f_k(\varphi_f)) = v_{k+1}$$

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A semiroot f_k is irreducible and $\Gamma_k = \Gamma_{f_k} = \langle \frac{v_0}{e_k}, \ldots, \frac{v_k}{e_k} \rangle$.

Semiroots

Given $f \in \mathbb{C}\{X\}[Y]$ irreducible with $\Gamma_f = \langle v_0, v_1, \ldots, v_g \rangle$.

A polynomial $f_k \in \mathbb{C}\{X\}[Y]$ for $0 \leq k < g$ is called a k-**semiroot** of f if f_k is monic,

$$deg(f_k) = \frac{v_0}{gcd(v_0, \ldots, v_k)} = \frac{v_0}{e_k} \text{ and } \nu_f(f_k) = ord_t(f_k(\varphi_f)) = v_{k+1}$$

A semiroot f_k is irreducible and $\Gamma_k = \Gamma_{f_k} = \langle \frac{v_0}{e_k}, \ldots, \frac{v_k}{e_k} \rangle$.

We have

$$e_k \Gamma_k \subset \Gamma_f \qquad (= e_k \Gamma_k + \langle v_{k+1}, \ldots, v_g \rangle)$$

$$\mu_f - 1 = e_k(\mu_k - 1) + \sum_{i=k+1}^{g} \left(\frac{e_{i-1}}{e_i} - 1 \right) v_i.$$

Questions:

Recall

$$e_k \Gamma_k \subset \Gamma_f \qquad \mu_f - 1 = e_k(\mu_k - 1) + \sum_{i=k+1}^{g} \left(\frac{e_{i-1}}{e_i} - 1 \right) v_i .$$

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What is the relation between τ_k and τ_f?

$e_k\Lambda_k \subset \Lambda_f$ can be false!

$\mathcal{E}(f_k) = \{AdX - BdY;\ A, B \in \mathbb{C}\{X\}[Y]\}$ $deg_Y(A) < deg_Y(f_k)$ and $deg_Y(B) < deg_Y(f_k) - 1$

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Proposition

If $\omega = AdX - BdY \in \mathcal{E}(f_k) \cap f_k \cdot \Omega(log\ \ C_k)$ *then*

$$\nu_f(\omega) = v_{k+1} - e_k\left(\mu_k - 1 + \frac{v_0}{e_k} - \nu_k(B)\right) \in \Lambda_f.$$

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Corollary

$$\nu_f\left(\mathcal{E}(f_k) \cap f_k \cdot \Omega(log\ \ C_k)\right) \ \cap\ \left(\Lambda_f \setminus \Gamma_f\right) = \\ = \{v_{k+1} - e_k\delta;\ \delta \in \mathbb{N}^* \setminus \Lambda_k \ or\ -\delta \in \mathbb{N} \setminus \Gamma_k\} =: M_k^0.$$

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Given $\delta_k \in \Lambda_k \setminus \Gamma_k$ we define

$\Theta_k(\delta_k) = \max\{\nu_k(B);\ \delta_k = \nu_k(\omega)$ and $\omega = AdX - BdY \in \mathcal{E}(f_k) \setminus f_k \cdot \Omega(log\ C_k)\}$.

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$$L_k^1 = \{\delta_k \in \Lambda_k \setminus \Gamma_k;\ e_k(\delta_k - \Theta_k(\delta_k)) < \beta_{k+1}\}$$
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Proposition

We have that $M_k^1 := e_k L_k^1 \subset \Lambda_f \setminus \Gamma_f$ *and*

$$M_k^2 := \left\{ v_{k+1} - e_k \left(\mu_k - 1 + \frac{v_0}{e_k} - \Theta_k(\delta_k) \right);\ \delta_k \in L_k^2 \right\} \subset \Lambda_f \setminus \Gamma_f.$$

Main results

Theorem

For any C_f and k-semiroot f_k, $0 \leq k < g$ we have

$$\left\{ \sum_{i=k+1}^{g} s_i v_i + m_k;\ \ m_k \in M_k^0 \right\} \ \dot{\cup}\ M_k^1 \ \dot{\cup}\ \left\{ \sum_{i=k+1}^{g} s_i v_i + m_k;\ \ m_k \in M_k^2 \right\} \subseteq \Lambda_f \setminus \Gamma_f$$

with $0 \leq s_i < \frac{e_{i-1}}{e_i};\ \ i = k+2, \ldots, g$ *and* $\ 0 \leq s_{k+1} \leq \frac{e_k}{e_{k+1}} - 2.$

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Any branch with $\Gamma = \langle 6, 9, 19 \rangle$ can be expressed by

$$(t^6, t^9 + t^{10} + a_{11} t^{11} + a_{14} t^{14} + a_{17} t^{17} + a_{20} t^{20} + a_{23} t^{23} + a_{26} t^{26}).$$

restrictions	$\Lambda \setminus \Gamma$
$a_{11} \notin \{-\frac{1}{2}, \frac{29}{18}\}$	$\{16, 22, 26, 29, 32, 35, 41\}$
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Consequence

Corollary

For any C_f *and* k*-semiroot* f_k, $0 \leq k < g$ *we have*

$$\tau_f \leq \mu_f - \mu_k - (e_k - 2e_{k+1})\tau_k.$$

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Corollary

For any branch C_f with semigroup $\Gamma_f = \langle v_0, \ldots, v_g \rangle$ we have

$$\tau_f \leq \mu_f - \frac{(3e_{g-1} - 2)}{4}\mu_{g-1}.$$

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Corollary

For any branch C_f with semigroup $\Gamma_f = \langle v_0, \ldots, v_g \rangle$ and $e_{g-1} = gcd(v_0, \ldots, v_{g-1}) = 2$ we have

$$\tau_f = \mu_f - \mu_{g-1}.$$

Thank you very much for your attention!