On the relative size of toric bases

Christos Tatakis
(joint work with A.Thoma)

Department of Mathematics
University of Ioannina, Greece

September 9, 2021

Definition of the Toric Ideal

Let $A = \{\mathbf{a}_1, \ldots, \mathbf{a}_m\} \subseteq \mathbb{N}^n$ be a vector configuration and $\mathbb{N}A := \{l_1\mathbf{a}_1 + \cdots + l_m\mathbf{a}_m \mid l_i \in \mathbb{N}\}$ the corresponding affine semigroup. We grade the polynomial ring $\mathbb{K}[x_1, \ldots, x_m]$ over an arbitrary field $\mathbb{K}$ by the semigroup $\mathbb{N}A$ setting $\deg_A(x_i) = \mathbf{a}_i$ for $i = 1, \ldots, m$. For $\mathbf{u} = (u_1, \ldots, u_m) \in \mathbb{N}^m$, we define the *A-degree* of the monomial $\mathbf{x}^{\mathbf{u}} := x_1^{u_1} \cdots x_m^{u_m}$ to be

$$\deg_A(\mathbf{x}^{\mathbf{u}}) = u_1\mathbf{a}_1 + \cdots + u_m\mathbf{a}_m \in \mathbb{N}A,$$

while the usual degree of $\mathbf{x}^{\mathbf{u}}$ is defined as $\deg(\mathbf{x}^{\mathbf{u}}) = u_1 + \cdots + u_m$.

Definition

The toric ideal I_A associated to A is the binomial ideal

$$I_A = < \mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}} : \ \deg_A(\mathbf{x}^{\mathbf{u}}) = \deg_A(\mathbf{x}^{\mathbf{v}}) > .$$

Elements of Graph Theory

Let G be a finite simple connected graph on the vertex set $V(G) = \{v_1, \ldots, v_n\}$ and $E(G) = \{e_1, \ldots, e_q\}$ be the set of edges of G.
A walk of length s connecting $v_1 \in V(G)$ and $v_{s+1} \in V(G)$ is a finite sequence of the form

$$w = (\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_s, v_{s+1}\})$$

with each $\{v_j, v_{j+1}\} \in E(G)$. An even (respectively odd) walk is a walk of even (respectively odd) length. A walk $w = (e_1 = \{v_1, v_2\}, e_2 = \{v_2, v_3\}, \ldots, e_q = \{v_s, v_{s+1}\})$ is called closed if $v_{s+1} = v_1$. A cycle is a closed walk

$$(\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_q, v_1\})$$

with $v_k \neq v_j$, for every $1 \leq k < j \leq q$.

Definition of I_G

Let G be a finite simple connected graph with vertices $V(G) = \{v_1, \dots, v_n\}$ and edges $E(G) = \{e_1, \dots, e_m\}$.

Let $\mathbb{K}[e_1, \dots, e_m]$ be the polynomial ring in the m variables $e_1, \dots, e_m$ over a field $\mathbb{K}$.

We will associate each edge $e = \{v_i, v_j\} \in E(G)$ with the element $a_e = v_i + v_j$ in the free abelian group $\mathbb{Z}^n$ with basis the set of vertices of G. Each vertex $v_j \in V(G)$ is associated with the vector $(0, \dots, 0, 1, 0, \dots, 0)$, where the nonzero component is in the j position.

Definition

We denote by I_G the toric ideal I_{A_G} in $\mathbb{K}[e_1, \dots, e_m]$, where $A_G = \{a_e \mid e \in E(G)\} \subset \mathbb{Z}^n$.

Definition of I_G

Given an even closed walk $w = (e_1, \ldots, e_{2q-1}, e_{2q})$ of the graph G we denote by B_w the binomial

$$B_w = \prod_{k=1}^{q} e_{2k-1} - \prod_{k=1}^{q} e_{2k} \in I_G.$$

Theorem (Villarreal, 1995)

The toric ideal I_G is generated by binomials of this form

$$I_G = < B_w, \text{ } w \text{ is an even closed walk of } G > .$$

Example

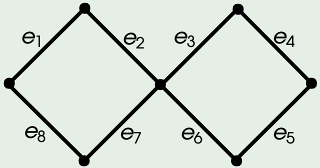


Figure: Example of I_G

$w_1 = (e_1, e_2, e_7, e_8)$ we have that $E^+(w_1) = e_1e_7$ and $E^-(w_1) = e_2e_8$ therefore $B_{w_1} = e_1e_7 - e_2e_8$.

$w_2 = (e_3, e_4, e_5, e_6) \Longrightarrow B_{w_2} = e_3e_5 - e_4e_6$

$w_3 = (e_1, e_2, \ldots, e_8) \Longrightarrow B_{w_3} = e_1e_3e_5e_7 - e_2e_4e_6e_8$.

Therefore

$$B_{w_1}, B_{w_2}, B_{w_3} \in I_G.$$

Graver basis

There are several sets for a toric ideal which include crucial information about it, such as the Graver basis, the Markov bases, the universal Gröbner basis, the universal Markov basis and the set of the circuits.

Definition (Graver basis)

An irreducible binomial $\mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}}$ in I_A is called primitive if there is no other binomial $\mathbf{x}^{\mathbf{w}} - \mathbf{x}^{\mathbf{z}}$ in I_A, such that $\mathbf{x}^{\mathbf{w}}$ divides $\mathbf{x}^{\mathbf{u}}$ and $\mathbf{x}^{\mathbf{z}}$ divides $\mathbf{x}^{\mathbf{v}}$. The set of the primitive binomials is finite, forms the Graver basis of I_A and is denoted by Gr_A.

Example

In the previous example the binomial $B_{w_3} = e_1e_3e_5e_7 - e_2e_4e_6e_8$ is not primitive because there exists the binomial $B_{w_2} = e_3e_5 - e_4e_6$ such that $e_3e_5 \mid e_1e_3e_5e_7$ and $e_4e_6 \mid e_2e_4e_6e_8$.

Markov bases

A set $\mathcal{A}$ of generators of I_G is minimal, if there is no other set $\mathcal{B}$ of generators of I_G such that $\mathcal{B} \subsetneq \mathcal{A}$. A generator of I_G is called minimal if it belongs in at least one minimal set of generators of I_G.

Example

In the previous example we saw that the ideal I_G has three generators:

$$B_{w_1} = e_1e_7 - e_2e_8, B_{w_2} = e_3e_5 - e_4e_6, B_{w_3} = e_1e_3e_5e_7 - e_2e_4e_6e_8.$$

Therefore

$$I_G = \langle B_{w_1}, B_{w_2}, B_{w_3} \rangle.$$

We remark that B_{w_3} is not minimal, since

$$B_{w_3} = e_3e_5B_{w_1} + e_2e_8B_{w_2}.$$

Thus $I_G = \langle B_{w_1}, B_{w_2} \rangle$.

Markov bases

Definition (Markov basis)

A minimal generating set of the ideal is called a Markov basis of I_A and is denoted by M_A.

Definition (Markov basis)

The universal Markov basis of the ideal I_A is denoted by $\mathcal{M}_A$ and is defined as the union of all the Markov bases of the ideal and is a finite set.

Example

In the previous example we saw that the for the ideal $I_G = \langle B_{w_1}, B_{w_2} \rangle$.
The ideal has exactly one Markov bases and it follows that
$M_A = \mathcal{M}_A = \{B_{w_1}, B_{w_2}\}$.

The set of the circuits of I_A

The support of a monomial $\mathbf{x}^\mathbf{u}$ of $\mathbb{K}[x_1, \ldots, x_m]$ is $\text{supp}(\mathbf{x}^\mathbf{u}) := \{i \mid x_i \text{ divides } \mathbf{x}^\mathbf{u}\}$ and the support of a binomial $B = \mathbf{x}^\mathbf{u} - \mathbf{x}^\mathbf{v}$ is $\text{supp}(B) := \text{supp}(\mathbf{x}^\mathbf{u}) \cup \text{supp}(\mathbf{x}^\mathbf{v})$.

Definition (the set of the circuits)

An irreducible nonzero binomial is called circuit if it has minimal support. The set of the circuits of a toric ideal I_A is denoted by $\mathcal{C}_A$.

Example

In the previous example we saw that the for the ideal I_G three of its elements are:

$B_{w_1} = e_1e_7 - e_2e_8, B_{w_2} = e_3e_5 - e_4e_6, B_{w_3} = e_1e_3e_5e_7 - e_2e_4e_6e_8.$

The corresponding supports are:

$\text{supp}(B_{w_1}) = \{1, 2, 7, 8\}, \text{supp}(B_{w_2}) = \{3, 4, 5, 6\}$ and $\text{supp}(B_{w_3}) = \{1, 2, 3, 4, 5, 6, 7, 8\}$.

We note that $\mathcal{C}_A = \{B_{w_1}, B_{w_2}\}$.

Universal Gröbner basis

Definition (universal Gröbner basis)

The universal Gröbner basis of an ideal I_A is defined as the union of all reduced Gröbner bases $G_<$ of I_A, as $<$ runs over all term orders. It is a finite subset of the I_A and it is a Gröbner basis for the ideal with respect to all admisible term orders. It is denoted by $\mathcal{U}_A$.

Example

In the previous example, easily we compute that the universal Gröbner basis of the corresponding ideal is $\mathcal{U}_A = \{B_{w_1}, B_{w_2}\}$.

A relation between some of the above sets was given by B. Sturmfels:

Theorem (B. Sturmfels)

For any toric ideal I_A it holds:

$$\mathcal{C}_A \subseteq \mathcal{U}_A \subseteq Gr_A.$$

What about the Markov bases of the ideal?

Every reduced Gröbner basis is a generating set of the toric ideal I_A consisting of binomials, therefore it contains also a Markov basis. Thus the universal Gröbner basis and the Graver basis contain at least one Markov basis.

It holds that $\mathcal{M}_A \subseteq Gr_A \iff \mathbb{N}A$ is positive $(\mathbb{N}A \cap (-\mathbb{N}A) = \{0\})$

It is well known that all the above inclusions may or may not be strict.

Important classes of ideals

In famous classes of ideals, the equality happens between some of the above bases and is combined with interesting geometric, combinatorial and homological properties.

Robust ideals are those ideals for which the universal Gröbner basis is a Markov basis ($M_A = \mathcal{U}_A$).

Generalized robust ideals are those ideals for which the universal Gröbner basis is also the universal Markov basis ($\mathcal{M}_A = \mathcal{U}_A$).

Strongly robust ideals are those ideals for which the Graver basis is a Markov basis ($M_A = Gr_A$).

Unimodular toric ideals are those ideals for which all the elements in the Graver basis are circuits ($\mathcal{C}_A = Gr_A$).

For example, lawrence toric ideals are strongly robust but also toric ideals of non pyramidal self dual projective toric varieties are strongly robust.

Two open problems

Problem I

If I_A is a robust toric ideal, is I_A minimally generated by its Graver basis?

Equivalently,

If $M_A = \mathcal{U}_A \Longrightarrow M_A = Gr_A$?

Problem II

If I_A is a generalized robust toric ideal, are the sets $\mathcal{M}_A$ and $\mathcal{U}_A$ equal?

Equivalently,

If $\mathcal{M}_A = \mathcal{U}_A \Longrightarrow \mathcal{M}_A = Gr_A$?

The problem I

As we presented it holds $\mathcal{C}_A \subseteq \mathcal{U}_A \subseteq Gr_A$. In the case of toric ideals of graphs, we know that every Markov basis of the ideal belongs also to its universal Gröbner basis, but this is not true in the general case. In general, we have no information about the differences of the sizes between the above sets. It is reasonable to ask about the comparison of the size of a Markov basis of the ideal with the subsets $\mathcal{C}_A, \mathcal{U}_A, Gr_A$ of I_A.

Problem I
Present theorems concerning bounds on the size of these bases in terms of the size of the other bases.

The problem II

One of the fundamental problems in toric algebra is to give good upper bounds on the degrees of the elements of the Graver basis. These bounds have important implications to:

- Integer programming
- Computational Algebraic Geometry
- Algebraic Statistics

There exist several bounds on the degrees of the elements of the Graver basis of a toric ideal, see for example.

The problem II

Since maximal degree of a set can be considered as a measure of the size of the corresponding basis, it is natural for someone to study about the maximal degrees of the elements of the above sets.

Conjecture (Sturmfels, (1995))

The circuits always have the maximal degree among the elements of the Graver basis.

Answer (S.Hosten, R.Thomas)

They gave a counterexample of a toric ideal such that the maximal degree of the elements of the Graver basis was 16 while the maximal degree of the circuits was 15.

The problem II

This counterexample (Hosten, Thomas) led B. Sturmfels to alter the conjecture to the following (known as true circuit conjecture):

Conjecture (Sturmfels, (1996))

the maximum degree of a circuit or the maximum *true* degree of a circuit bounds the maximum degree of any Graver basis element

Answer (-, A.Thoma (2011))

We gave an infinite family of counterexamples of toric ideals and elements of the Graver basis for which their maximal degrees are not bounded above polynomially by the corresponding maximal degree or the maximal true degree of a circuit.

Problem II

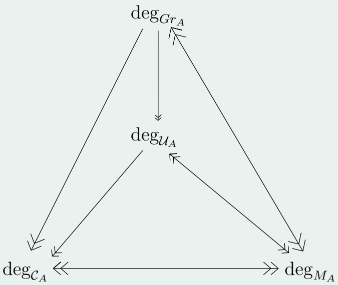
Present theorems concerning bounds on the maximal degree of the elements of these bases in terms of the maximal degree of the elements of the other bases.

Results: Problem II

For the maximal degree of the elements of the sets: $\mathcal{C}_A, \mathcal{U}_A, M_A, Gr_A$:

Theorem (-, Thoma (2021))

$B \twoheadrightarrow C$ represents that the maximal degree of the elements of the set B cannot be bounded above by a polynomial on the maximal degree of the elements of C.

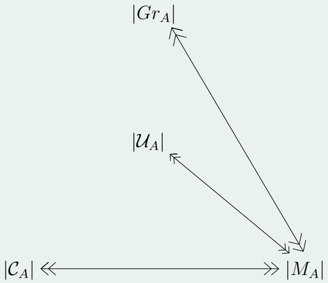


Results: Problem I

For the size of the sets: $\mathcal{C}_A, \mathcal{U}_A, M_A, Gr_A$ we were able to provide the corresponding theorems in all cases except three

Theorem (-, Thoma (2021))

$B \twoheadrightarrow C$ represents that the size of the set B cannot be bounded above by a polynomial on the size of the set of C.



Open problem

Problem

Complete the previous theorem (on the size of toric bases).

Thank you!!!