On robustness and related properties on toric ideals

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Definition of the toric ideal

Let $A=\{\mathbf{a}_1,\ldots,\mathbf{a}_m\}\subseteq\mathbb{N}^n$ be a vector configuration and $\mathbb{N}A:=\{l_1\mathbf{a}_1+\cdots+l_m\mathbf{a}_m \mid l_i\in\mathbb{N}\}$ the corresponding affine semigroup. We grade the polynomial ring $\mathbb{K}[x_1,\ldots,x_m]$ over an arbitrary field $\mathbb{K}$ by the semigroup $\mathbb{N}A$ setting $\deg_A(x_i)=\mathbf{a}_i$ for $i=1,\ldots,m$. For $\mathbf{u}=(u_1,\ldots,u_m)\in\mathbb{N}^m$, we define the *A-degree* of the monomial $\mathbf{x}^{\mathbf{u}}:=x_1^{u_1}\cdots x_m^{u_m}$ to be

$$\deg_A(\mathbf{x}^{\mathbf{u}})=u_1\mathbf{a}_1+\cdots+u_m\mathbf{a}_m\in\mathbb{N}A,$$

while the usual degree of $\mathbf{x}^{\mathbf{u}}$ is defined as $\deg(\mathbf{x}^{\mathbf{u}})=u_1+\cdots+u_m$.

Definition

The toric ideal I_A associated to A is the binomial ideal

$$I_A=<\mathbf{x}^{\mathbf{u}}-\mathbf{x}^{\mathbf{v}}:\ \deg_A(\mathbf{x}^{\mathbf{u}})=\deg_A(\mathbf{x}^{\mathbf{v}})>.$$

Example of a toric ideal

Example

We consider the following set:
$A = \{(2,1,0),(1,2,0),(0,2,1),(0,1,2),(1,0,2),(2,0,1)\} \subseteq \mathbb{N}^3$.

By CoCoA, we compute the corresponding toric ideal:

$$\mathcal{I}_A = \langle x_2x_5 - x_3x_6, x_1x_4 - x_3x_6, x_1x_3^2 - x_2^2x_4, x_2x_4^2 - x_3^2x_5, x_1^2x_3 - x_2^2x_6,$$

$$x_1^2x_5 - x_2x_6^2, x_3x_5^2 - x_4^2x_6, x_1x_5^2 - x_4x_6^2, x_1x_3x_5 - x_2x_4x_6\rangle.$$

For example, we consider the minimal generator $x_1^2x_5 - x_2x_6^2$. Then we have :

$$\deg_A(x_1^2x_5) = 2\cdot(2,1,0) + 1\cdot(1,0,2) = (5,2,2)$$

and

$$\deg_A(x_2x_6^2) = 1\cdot(1,2,0) + 2\cdot(2,0,1) = (5,2,2).$$

History of toric ideals

Toric ideals started to studying:

- M.Hochster, Rings of invariants of toric, Cohen-Macaulay rings generated by monomials and polytopes, Annals of Math. 96 318-337 (1972)
- W. Fulton, Lectures in Geometry and Algebraic Geometry Conference, Washington University, MI, USA (1989)

The following 3 papers were very important for the chapter of toric algebra

- A.Simis, W.Vasconcelos, R.Villareal, On the ideal theory of graphs, J.Algebra 167 (1994), 389-416.
- D.Eisenbud, B.Sturmfels, Binomial ideals, Duke Math. J. 84 (1996), 1-45.
- P.Diaconis, B.Sturmfels, Algebraic algorithms for sampling from conditional distributions, Annals of Statist. 26 (1998), 363-397.

Definition of a toric ideal of a graph I_G

Let G be a finite simple connected graph with vertices $V(G) = \{v_1, \dots, v_n\}$ and edges $E(G) = \{e_1, \dots, e_m\}$.

Let $\mathbb{K}[e_1, \dots, e_m]$ be the polynomial ring in the m variables $e_1, \dots, e_m$ over a field $\mathbb{K}$.

We will associate each edge $e = \{v_i, v_j\} \in E(G)$ with the element $a_e = v_i + v_j$ in the free abelian group $\mathbb{Z}^n$ with basis the set of vertices of G. Each vertex $v_j \in V(G)$ is associated with the vector $(0, \dots, 0, 1, 0, \dots, 0)$, where the nonzero component is in the j position.

Definition

We denote by I_G the toric ideal I_{A_G} in $\mathbb{K}[e_1, \dots, e_m]$, where $A_G = \{a_e \mid e \in E(G)\} \subset \mathbb{Z}^n$.

Definition of I_G

Given an even closed walk $w = (e_1, \dots, e_{2q-1}, e_{2q})$ of the graph G we denote by B_w the binomial

$$B_w = \prod_{k=1}^{q} e_{2k-1} - \prod_{k=1}^{q} e_{2k} \in I_G.$$

Theorem (Villarreal, 1995)

The toric ideal I_G is generated by binomials of this form

$$I_G = < B_w,\ w \text{ is an even closed walk of } G > .$$

Example

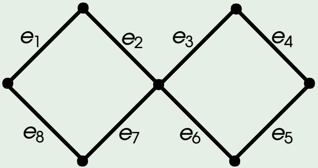


Figure: Example of I_G

- $w_1 = (e_1, e_2, e_7, e_8)$ we have that $E^+(w_1) = e_1e_7$ and $E^-(w_1) = e_2e_8$ therefore $B_{w_1} = e_1e_7 - e_2e_8$.
- $w_2 = (e_3, e_4, e_5, e_6) \implies B_{w_2} = e_3e_5 - e_4e_6$
- $w_3 = (e_1, e_2, \ldots, e_8) \implies B_{w_3} = e_1e_3e_5e_7 - e_2e_4e_6e_8$.

Therefore

$$B_{w_1}, B_{w_2}, B_{w_3} \in I_G.$$

Graver basis

There are several sets for a toric ideal which include crucial information about it, such as the Graver basis, the Markov bases, the universal Gröbner basis, the universal Markov basis and the set of the circuits.

Definition (Graver basis)

An irreducible binomial $\mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}}$ in I_A is called primitive if there is no other binomial $\mathbf{x}^{\mathbf{w}} - \mathbf{x}^{\mathbf{z}}$ in I_A, such that $\mathbf{x}^{\mathbf{w}}$ divides $\mathbf{x}^{\mathbf{u}}$ and $\mathbf{x}^{\mathbf{z}}$ divides $\mathbf{x}^{\mathbf{v}}$. The set of the primitive binomials is finite, forms the Graver basis of I_A and is denoted by Gr_A.

Example

In the previous example the binomial $B_{w_3} = e_1e_3e_5e_7 - e_2e_4e_6e_8$ is not primitive because there exists the binomial $B_{w_2} = e_3e_5 - e_4e_6$ such that $e_3e_5 \mid e_1e_3e_5e_7$ and $e_4e_6 \mid e_2e_4e_6e_8$.

Graver basis - importance

The Graver basis of a toric ideal, is a very important set and has a lot of information for the ideal:

1. It is a set of generators of the ideal
2. It includes at least one minimal set of generators of the ideal
3. If $\mathbb{N}A$ is pointed, then every minimal generator of the ideal belongs to its Graver basis
4. Every reduced Gröbner basis is a subset of the Graver basis
5. The universal Gröbner basis of the ideal is a subset of the Graver basis
6. The set of the circuits of the ideal is a subset of the Graver basis

Markov bases

A set $\mathcal{A}$ of generators of I_G is minimal, if there is no other set $\mathcal{B}$ of generators of I_G such that $\mathcal{B} \subsetneq \mathcal{A}$. A generator of I_G is called minimal if it belongs in at least one minimal set of generators of I_G.

Example

In the previous example we saw that the ideal I_G has three generators:

$$B_{w_1} = e_1e_7 - e_2e_8, B_{w_2} = e_3e_5 - e_4e_6, B_{w_3} = e_1e_3e_5e_7 - e_2e_4e_6e_8.$$

Therefore

$$I_G = \langle B_{w_1}, B_{w_2}, B_{w_3} \rangle.$$

We remark that B_{w_3} is not minimal, since

$$B_{w_3} = e_3e_5B_{w_1} + e_2e_8B_{w_2}.$$

Thus $I_G = \langle B_{w_1}, B_{w_2} \rangle$.

Markov bases

Definition (Diaconis-Sturmfels, Annals of Statistics (1998))

A minimal generating set of the ideal is called a Markov basis of I_A and is denoted by M_A.

Definition (Universal Markov basis)

The universal Markov basis of the ideal I_A is denoted by $\mathcal{M}_A$ and is defined as the union of all the Markov bases of the ideal and is a finite set.

Example

In the previous example we saw that the for the ideal $I_G = \langle B_{w_1}, B_{w_2} \rangle$. The ideal has exactly one Markov bases and it follows that $M_A = \mathcal{M}_A = \{B_{w_1}, B_{w_2}\}$.

The set of the circuits of I_A

The support of a monomial $\mathbf{x}^\mathbf{u}$ of $\mathbb{K}[x_1,\dots,x_m]$ is $\text{supp}(\mathbf{x}^\mathbf{u}) := \{i \mid x_i \text{ divides } \mathbf{x}^\mathbf{u}\}$ and the support of a binomial $B = \mathbf{x}^\mathbf{u} - \mathbf{x}^\mathbf{v}$ is $\text{supp}(B) := \text{supp}(\mathbf{x}^\mathbf{u}) \cup \text{supp}(\mathbf{x}^\mathbf{v})$.

Definition (the set of the circuits)

An irreducible nonzero binomial is called circuit if it has minimal support. The set of the circuits of a toric ideal I_A is denoted by $\mathcal{C}_A$.

Example

In the previous example we saw that the for the ideal I_G three of its elements are:
$B_{w_1} = e_1e_7 - e_2e_8$, $B_{w_2} = e_3e_5 - e_4e_6$, $B_{w_3} = e_1e_3e_5e_7 - e_2e_4e_6e_8$.
The corresponding supports are:
$\text{supp}(B_{w_1}) = \{1,2,7,8\}$, $\text{supp}(B_{w_2}) = \{3,4,5,6\}$ and
$\text{supp}(B_{w_3}) = \{1,2,3,4,5,6,7,8\}$.
We note that $\mathcal{C}_A = \{B_{w_1}, B_{w_2}\}$.

Universal Gröbner basis

Definition (universal Gröbner basis)

The universal Gröbner basis of an ideal I_A is defined as the union of all reduced Gröbner bases $G_<$ of I_A, as $<$ runs over all term orders. It is a finite subset of the I_A and it is a Gröbner basis for the ideal with respect to all admisible term orders. It is denoted by $\mathcal{U}_A$.

Example

In the previous example, easily we compute that the universal Gröbner basis of the corresponding ideal is $\mathcal{U}_A = \{B_{w_1}, B_{w_2}\}$.

Theorem (B. Sturmfels)

For any toric ideal I_A it holds:

$$\mathcal{C}_A \subseteq \mathcal{U}_A \subseteq Gr_A.$$

Also:

Theorem (Charalambous, Thoma, Vladoiu (2015))

For any toric ideal I_A where A defines a pointed semigroup, it holds:

$$\mathcal{M}_A \subseteq Gr_A.$$

Theorem (- (2016))

For any toric ideal I_G of a graph G, it holds:

$$\mathcal{M}_{A_G} \subseteq \mathcal{U}_{A_G}.$$

Important classes of ideals

In famous classes of ideals, the equality happens between some of the above bases and is combined with interesting geometric, combinatorial and homological properties.

Definition (Boocher and Robeva, (2015))

A toric ideal is called **robust** if its universal Gröbner basis is a minimal set of generators (i.e. $M_A = \mathcal{U}_A$)

Definition (- , (2016))

A toric ideal is called **generalized robust** if its universal Gröbner basis equals its universal Markov basis (i.e. $\mathcal{M}_A = \mathcal{U}_A$)

Definition (Sullivant, (2019))

A toric ideal is called **strongly robust** if its Graver basis equals with its set of indispensable binomials (i.e. $M_A = Gr_A$)

Quadratic toric ideals

Definition (Quadratic toric ideal)

A toric ideal is called **quadratic** if it is generated by quadrics, i.e. its minimal generators are of degree two.

Example

In the previous example we saw that:

$$I_G = \langle B_{w_1} = e_1 e_7 - e_2 e_8, B_{w_2} = e_3 e_5 - e_4 e_6 \rangle.$$

Therefore the ideal is a quadratic ideal.

Problems I - II

Problem I
Characterize the graphs giving rise to generalized robust toric ideals such that they are generated by quadrics.

Problem II
Characterize the graphs giving rise to robust toric ideals such that they are generated by quadrics.

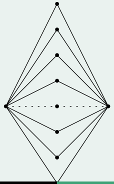
Previous results and the interest

Robustness (and generalized robustness) is a rare property and there are very few nontrivial examples of robust ideals. Previous results:

1. Ohsugi and Hibi (1999) characterized completely the graphs whose toric ideals are generated by quadrics.
2. Boocher and Robeva (2015) studying robustness for toric ideals which are generated by quadrics.
3. Boocher et.all (2015) studying robustness for toric ideals of graphs:

Theorem (Boocher et.all (2015)

All robust ideals generated by quadrics are graph ideals and come from the family of disjoint unions of graphs given following:



Previous results and the interest

1. White's conjecture (1980): the toric ideal I_M for a matroid M is generated by quadrics.
2. Some of the interests in studying robustness stems from the fact that they are ideals which are minimally generated by a Gröbner basis.
3. Whenever I is an ideal with a quadratic Gröbner basis, then $\mathbb{K}[x_1, \ldots, x_n]$ is a Koszul algebra (since the ideals provide examples of Koszul algebras).

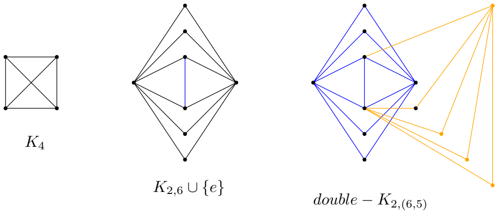
Results

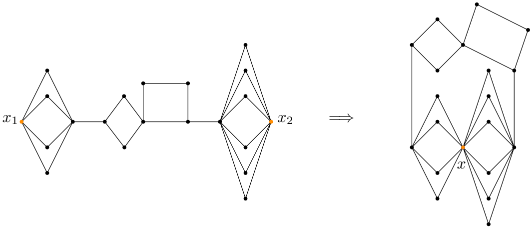
The next theorem gives a nice property which motivates the study of generalized robust toric ideals which are generated by quadrics.

Theorem (I.Garcia-Marco, -)

A homogeneous toric ideal $I_A \subseteq \mathbb{K}[x_1, \ldots, x_m]$ is generalized robust and generated by quadrics if and only if the universal Gröbner basis of I_A only consists of quadrics.

Answering Problems I-II





The construction of a necklace $K_{2,I}$ graph

Answering Problems I-II

We present the first main result which answers Problem I, in which we give a complete characterization of the graphs G such that I_G is a generalized robust toric ideal generated by quadratic binomials.

Theorem (I.Garcia-Marco, -)

Let G be a non bipartite graph. The ideal is generalized robust and is generated by quadrics if and only if all the blocks of G are bipartite except one, which is either a K_4 or a $K_{2,\ell} \cup \{e\}$ or a double-$K_{2,(r,s)}$ or a necklace-$K_{2,\ell}$ graph. Every bipartite block of the graph G are of type $K_{2,\ell}$ for some ℓ or cut edges.

Answering Problems I-II

We present the second main result which answers Problem II, in which we give a complete characterization of the graphs G such that I_G is a robust toric ideal generated by quadratic binomials.

Theorem (I.Garcia-Marco, -)

Let G be a non bipartite graph. The ideal is robust and is generated by quadrics if and only if all the blocks of G are bipartite except one, which is a $K_{2,\ell} \cup \{e\}$ *or a double-*$K_{2,(r,s)}$ *or a necklace-*$K_{2,\ell}$ *graph. Every bipartite block of the graph G are of type* $K_{2,\ell}$ *for some* ℓ *or cut edges.*

Remark: For toric ideals of graphs we know that the notion of strongly robust and robust coincide. Therefore the above theorem, also characterizes completely the strongly robust toric ideals of graphs which are generated by quadrics.

Example I

Example

We present two examples:
Figure 1: A graph G such that the corresponding toric ideal I_G is generalized robust and is generated by quadrics but it is not robust and it is not strongly robust.
Figure 2: A graph G such that the corresponding toric ideal I_G is both a robust, a generalized robust and a strongly robust and is generated by quadrics.

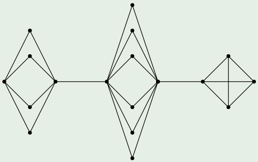


Figure 1

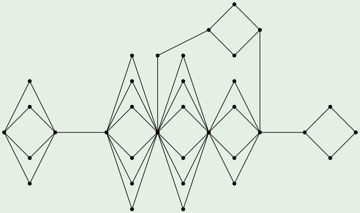
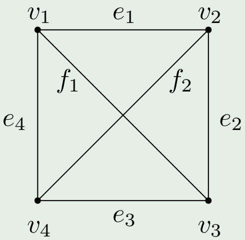


Figure 2

Example II

Example

Consider the complete graph on four vertices K_4 on the vertex set $V(K_4) = \{v_1, v_2, v_3, v_4\}$ and on the edge set $E(K_4) = \{e_1 = \{v_1, v_2\}, e_2 = \{v_2, v_3\}, e_3 = \{v_3, v_4\}, e_4 = \{v_4, v_1\}, f_1 = \{v_1, v_3\}, f_2 = \{v_2, v_4\}\}$.



It is clear that we have exactly three 4-cycles $w_1 = (v_1, v_2, v_3, v_4, v_1)$, $w_2 = (v_1, v_2, v_4, v_3, v_1)$ and $w_3 = (v_1, v_4, v_2, v_3, v_1)$.

Example II

Example

The corresponding ideal I_{K_4} is generated by the three binomials:

$$I_{K_4} = \langle B_{w_1} = e_1e_3 - e_2e_4, B_{w_2} = e_1e_3 - f_1f_2, B_{w_3} = e_2e_4 - f_1f_2 \rangle.$$

Obviously, none of $B_{w_1}, B_{w_2}, B_{w_3}$ is indispensable since

$$B_{w_i} \in \langle B_{w_j}, B_{w_k} \rangle, \text{ for all distinct } i, j, k \text{ where } i, j, k \in \{1, 2, 3\}.$$

Thus, the ideal has three different Markov bases;

$$M_1 = \{B_{w_1}, B_{w_2}\}, M_2 = \{B_{w_1}, B_{w_3}\}, M_3 = \{B_{w_2}, B_{w_3}\}.$$

The universal Markov basis of the ideal is $\mathcal{M}_{K_4} = \{B_{w_1}, B_{w_2}, B_{w_3}\}$.
It is easy to check that the universal Gröbner basis of the ideal I_{K_4} is $\mathcal{U}_{K_4} = \{B_{w_1}, B_{w_2}, B_{w_3}\}$.
It follows that the quadratic ideal I_{K_4} is generalized robust but not robust.

Basic notions

Given a submonoid $\mathcal{S}$ of $(\mathbb{N}, +)$, then it has a unique minimal set of generators $A = \{a_1, \ldots, a_m\}$, and the toric ideal of $\mathcal{S}$ is defined as $I_{\mathcal{S}} := I_A$. Taking $d := \gcd(a_1, \ldots, a_m)$ and $A' := \{a_1/d, \ldots, a_m/d\}$, then $I_A = I_{A'}$. Hence, one may assume without loss of generality that $A = \{a_1, \ldots, a_m\}$ consists of relatively prime positive integers and, in this case, $\mathcal{S}$ is called a *numerical semigroup* .

By Krull's dimension theorem, any set of generators of an ideal $J \subseteq \mathbb{K}[\mathbf{x}]$ has at least $\mathrm{ht}(J)$ elements and is a complete intersection when equality occurs. Whenever $\mathcal{S} = \langle a_1, \ldots, a_m \rangle$ is a numerical semigroup, then $I_{\mathcal{S}}$ has height $m - 1$ and $I_{\mathcal{S}}$ is a complete intersection if and only if $\mu(I_{\mathcal{S}}) = m - 1$ or, in other words, if it can be generated by a set of $m - 1$ binomials (i.e. one of its Markov basis (and, thus, all its Markov bases) consists of $m - 1$ binomials).

Basic notions

Definition

A numerical semigroup $\mathcal{S}$ with minimal generating set $A = \{a_1, \ldots, a_m\}$ is said to be *free for the arrangement* $a_1, \ldots, a_m$ if

$$\text{lcm}(a_i, \gcd(a_{i+1}, \ldots, a_m)) \in \langle a_{i+1}, \ldots, a_m \rangle, \forall i \in \{1, \ldots, m-1\}.$$

We say that $\mathcal{S}$ is *free* if it is free for an arrangement of its minimal generating set.

Basic notions

Example

Consider the numerical semigroup $\mathcal{S} = \langle a_1, a_2, a_3, a_4 \rangle$ with $a_1 = 8,\ a_2 = 9,\ a_3 = 10,\ a_4 = 12$. We have that $\mathcal{S}$ is not free for the arrangement a_1, a_2, a_3, a_4 because $\operatorname{lcm}(a_1, \gcd(a_2, a_3, a_4)) = 8 \notin \langle a_2, a_3, a_4 \rangle$. However, $\mathcal{S}$ is free for the arrangement $a_2 = 9,\ a_3 = 10,\ a_1 = 8,\ a_4 = 12$. Indeed,

- $\operatorname{lcm}(a_2, \gcd(a_1, a_3, a_4)) = 18 = a_1 + a_3 \in \langle a_1, a_3, a_4 \rangle$,
- $\operatorname{lcm}(a_3, \gcd(a_1, a_4)) = 20 = a_1 + a_4 \in \langle a_1, a_4 \rangle$, and
- $\operatorname{lcm}(a_1, a_4) = 24 = 2a_4 \in \langle a_4 \rangle$.

Thus, $\mathcal{S}$ is a free numerical semigroup.

Results

Trying to connect the above notions with the robustness property, we proved the following theorem:

Theorem (I.Garcia-Marco, -)

Let $\mathcal{S}$ be a numerical semigroup. Then, $\mathcal{S}$ is free if and only if it has a Gröbner basis with $m-1$ elements.

Example

We consider the numerical semigroup $\mathcal{S} = \langle 9, 10, 8, 12 \rangle$ (as the previous example), which we saw that it is free for the arrangement
$a_2 = 9,\ a_3 = 10,\ a_1 = 8,\ a_4 = 12.$
By the above theorem, it follows that the reduced Gröbner basis with respect to the lexicographic order with $x_2 > x_3 > x_1 > x_4$ has 3 elements. Indeed, one can checks that it is

$$\{x_2^2 - x_1x_3, x_3^2 - x_1x_4, x_1^3 - x_4^2\}$$

Example

We consider the numerical semigroup $\mathcal{S} = \langle a_1, a_2, a_3, a_4 \rangle$ with
$a_1 = 10,\ a_2 = 14,\ a_3 = 15,\ a_4 = 21.$
We can compute that $\mathcal{S}$ is not free and $I_{\mathcal{S}}$ is a complete intersection.
By the above theorem, it follows that the ideal $I_{\mathcal{S}}$ cannot be minimally generated by a Gröbner basis.

Garcia-Sanchez, Ojeda and Rosales studied a family of affine submonoids of $\mathbb{N}^n$ which they called *semigroups with a unique Betti element*

Definition (Garcia-Sanchez, Ojeda and Rosales, (2012))

An affine monoid $\mathcal{S}$ with minimal generating set A has a unique Betti element if and only if the A-degrees of all the binomials in a Markov basis of I_A coincide.

Next, they characterized semigroups with a unique Betti element as those where $\mathcal{C}_A = Gr_A$ (i.e. the set of the circuits of I_A coincides with the Graver basis).

Moreover, in this family of semigroups one has that $\mathcal{C}_A \subseteq \mathcal{M}_A$. As a consequence of these two facts we have that:

Proposition

Every affine monoid with a unique Betti element defines a generalized robust toric ideal.

Problem III

What about the converse? In general, the converse statement of the above result is not true (for example, in toric ideals of graphs someone can easily constructs a graph G such that its ideal is generalized robust but it has not a unique Betti element).

Our second main result shows that the converse holds for numerical semigroups:

Theorem (I.Garcia-Marco, -)

A numerical semigroup defines a generalized robust toric ideal if and only if it has a unique Betti element.

Since numerical semigroups with a unique Betti element only have a unique minimal set of generators when A has two elements, we directly get the following:

Theorem (I.Garcia-Marco, -)

A numerical semigroup $\mathcal{S}$ defines a robust toric ideal if and only if $\mathcal{S}$ is 2-generated.

Open problems

We have that robustness property is preserved under an elimination of variables. However, we do not know if the same result is true when we replace robustness by generalized robustness.

Problem 1: Let I_A be a generalized robust toric ideal and $A' \subseteq A$, is $I_{A'}$ generalized robust?

We characterize when the toric ideal of a numerical semigroup has a complete intersection initial ideal. It would be interesting to seek the answer to the same question for toric ideals of graphs.

Problem 2: Characterize when the toric ideal I_G of a graph G has a complete intersection initial ideal.

Open problems

In our first main result in semigroups, we proved that free numerical semigroups have an initial ideal such that $\mu(I_S) = \mu(\mathrm{in}_{\prec}(I_S))$. There are further families of numerical semigroups with the same property. It is interesting to characterize completely the numerical semigroups with the above property.

Problem 3: Characterize the numerical semigroups such that

$$\mu(I_S) = \mu(\mathrm{in}_{\prec}(I_S))$$

for a monomial order $\prec$.

We have verified when the equality $\mathcal{M}_{I_S} = \mathcal{U}_{I_S}$ occurs for a numerical semigroup $\mathcal{S}$. It would be interesting to characterize when equality or containment of other toric bases holds.

Problem 4: Characterize the numerical semigroups $\mathcal{S}$ such that I_S is a circuit ideal.

Proving the above theorems...

Thank you!!!