Algebraic methods in an alleged proof of the Jacobian Conjecture

arXiv: 2306.03996

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(joint thoughts with dr Szymon Brzostowski)

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Introduction

The article:

Wolfgang Bartenwerfer, *The Cremona problem in dimension 2.* Archiv der Mathematik (Basel) 119(2022), 53 -62.

MR1099945 - Die Beschränktheit der Stückzahl der Fasern K-analytischer Abbildungen Reviewed

The boundedness of the number of fibers of K-analytic mappings

8 citations

Bartenwerfer, Wolfgang

MSC 32P05

J. Reine Angew. Math. **416** (1991), 49–70.

Article

MR0869753 - Fortsetzung kohärenter Garben an krummlinigen Hartogsfiguren und solchen mit beliebiger Basis Reviewed

1 citations

Extension of coherent sheaves on curvilinear Hartogs figures and on those with arbitrary base

MSC 32D25

Bartenwerfer, Wolfgang

Nederl. Akad. Wetensch. Indag. Math. **48** (1986), no. 4, 361–377.

MR0782387 - Zur Existenz einer Steinschen Umgebung eines abgeschlossenen Steinschen Unterraums Reviewed

MSC 32K10

On the existence of a Stein neighborhood of a closed Stein subspace

Article

Bartenwerfer, Wolfgang

Compositio Math. **54** (1985), no. 1, 79–93.

MR0653459 - Die strengen metrischen Kohomologiegruppen des Einheitspolyzylinders verschwinden Reviewed

3 citations

The strict metric cohomology groups of the unit polycylinder vanish

MSC 12J25

Bartenwerfer, Wolfgang

Nederl. Akad. Wetensch. Indag. Math. **44** (1982), no. 1, 101–106.

MR0622355 - k-holomorphe Vektorraumbündel auf offenen Polyzylindern Reviewed

k-holomorphic vector space bundles on open polycylinders

3 citations

Bartenwerfer, Wolfgang

MSC 32L05

J. Reine Angew. Math. **326** (1981), 214–220.

Article

Introduction

Review in Zentralblatt:

Bartenwerfer, Wolfgang
The Cremona problem in dimension 2. (English) Zbl 1501.14001
Arch. Math. 119, No. 1, 53-62 (2022); correction ibid. 120, No. 6, 665-666 (2023).

The famous Jacobian Conjecture claims that a polynomial mapping $F : \mathbb{C}^n \to \mathbb{C}^n$ with the constant non-zero jacobian ($\det \mathrm{Jac} F = \mathrm{const} \neq 0$) is a polynomial automorphism. The author proves this conjecture for $n = 2$. We put $F = (f, g)$. However, there are some weak points in the proof:

1. (communicated to me by Z. Jelonek and A. Parusinski). In Corollary 1.5 from the proportions $\frac{m}{n} = \frac{m_1}{n_1} = \frac{m_2}{n_2}$ the author infers that $|m_1 - m_2| \neq |n_1 - n_2|$. The second possible case $|m_1 - m_2| = |n_1 - n_2| = 0$ is omitted, which should be considered separately.
2. (communicated to me by Sz. Brzostowski). In Proposition 2.3 the cardinality of the fibers of $(f, g)|_V$ is calculated. It is equal to the sum of cardinalities of the fibers of the mappings: $(f, g^{1/k})|_V$, $(f, \varepsilon g^{1/k})|_V, \dots, (f, \varepsilon^{k-1} g^{1/k})|_V$, where ε is a k-th primitive root of 1 and $g^{1/k}$ is a fixed k-th root of g (under assumptions in the proof this root exists). Precisely, if $(x_0, y_0) \in V$ and we put $(\alpha, \beta) := (f(x_0, y_0), g(x_0, y_0))$, $\widetilde{\beta} := g^{1/k}(x_0, y_0)$ then

$$\#((f,g)|_V)^{-1}(\alpha, \beta) = \sum_{i=0}^{k-1} \#((f, \varepsilon^i g^{1/k})|_V)^{-1}(\alpha, \widetilde{\beta}).$$

The author claims that each term in the sum is the same and equal to $|n_1 - n_2|/k$. This is not justified because we don't know if the point $(\alpha, \widetilde{\beta})$ is in the images $(f, \varepsilon^i g^{1/k})(V)$ for $i = 1, \dots, k-1$ (this is true for $i = 0$).

Editorial remark: See also the correction [*R. Chill* and *G. Nebe*, Arch. Math. 120, No. 6, 665–666 (2023; Zbl 1512.14035)].

Reviewer: Tadeusz Krasiński (Łódź)

Introduction

Review in Mathematical Reviews:

Bartenwerfer, Wolfgang

The Cremona problem in dimension 2. (English summary)

Arch. Math. (Basel) **119** (2022), no. 1, 53–62.

See correction in: MR4598549

Classifications

14R15 - Jacobian problem

14G22 - Rigid analytic geometry

Citations

From References: 1

From Reviews: 0

Review

This paper contains an attempt to solve the Jacobian Conjecture in the two-dimensional case. Unfortunately, the claim of Proposition 2.3, the key proposition of the paper, is wrong. The author discovered a mistake in the proof of this proposition and will be withdrawing the paper.

Reviewer: Makar-Limanov, L. G.

Introduction

Correction to: Arch. Math. (2022) 119:53–62
https://doi.org/10.1007/s00013-022-01733-1

The proof of the main result of the original article [1] is wrong.

The original article claims to prove the Jacobian conjecture in dimension 2. After publication, the author and later the editors learned from Makar-Limanov that the method of proof cannot work as claimed. In an email to Makar-Limanov from October 31st 2022, the author admitted:

"*You are right that Proposition 2.3 must be wrong. It took me a certain time to find out, where the fault is located. It is hidden at the very end of the proof of Proposition 2.3:*

A fibre of $(f - G, g^{1/k})_{|V}$ *gives rise to a fibre of* $(f, g^{1/k})_{|V}$ *with the same cardinality, but the thing does not work "vice versa". So my argument is not valid. Consequently there will come out an "erratum", which comes up to a withdrawal of the whole paper.*

Sincerely Yours, Wolfgang Bartenwerfer"

As we did not receive the promised erratum by the deadline December 31st 2022, the editors in chief of the Archiv der Mathematik decided to publish this note as an erratum.

Jacobian Conjecture (Keller Conjecture, Cremona Problem)

Jacobian Conjecture (JC). If $(f,g): \mathbb{C}^2 \to \mathbb{C}^2$ is a polynomial mapping with jacobian equals identically 1, then (f,g) is a global polynomial automorphism of $\mathbb{C}^2$.

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Remark 2. JC is false for entire functions and in characteristic p.

Remark 3. JC is false over reals $\mathbb{R}$ under the weaker assumption that the jacobian nowhere vanish.

Remark 4. There are a lot of equivalent formulations of JC.

Jacobian Conjecture

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Remark 7. Information on JC:

- A. Essen, *Polynomial Automorphisms.* Birkhäuser 2000.

- A. Essen, S. Kuroda, A. Crachiola, *Polynomial Automorphisms and the Jacobian Conjecture.* Birkhäuser 2021.

„Proof" of Bartenwerfer.

Jacobian Conjecture.

Assumptions. $(f,g)\colon \mathbb{C}^2 \to \mathbb{C}^2$, $f,g \in \mathbb{C}[X,Y]$, $Jac(f,g) = 1$.

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Proof (reductio ad absurdum). Assume there exists a jacobian pair (f,g) which is not an automorphism. Since (f,g) is not an automorphism, by some Abhyankar theorems and jacobian condition ($Jac(f,g) = 1$) it follows that we may assume f and g have particular forms:

„Proof" of Bartenwerfer.

$$f(X,Y) = f_m(X,Y) + \cdots + f_{m'}(X,Y), \qquad m = \deg f > 0$$

$$g(X,Y) = g_n(X,Y) + \cdots + g_{n'}(X,Y), \qquad n = \deg g > 0$$

$$f_m(X,Y) = X^{m_1}Y^{m_2}, \qquad m_1 > 0, m_2 > 0, \ m_1 + m_2 = m$$

$$g_n(X,Y) = X^{n_1}Y^{n_2}, \qquad n_1 > 0, n_2 > 0, \ n_1 + n_2 = n,$$

and

$$\frac{m}{n} = \frac{m_1}{n_1} = \frac{m_2}{n_2} =: r \in \mathbb{Q}.$$

„Proof" of Bartenwerfer.

Additionally, we may assume $m \neq n$. In fact, if $m = n$, then by the above proportions

$$\frac{m}{n} = \frac{m_1}{n_1} = \frac{m_2}{n_2}$$

we get $m_1 = n_1$ i $m_2 = n_2$ and then we replace the pair (f, g) by new jacobian pair $(f - g, g)$, which fullfils this condition:

$$\deg(f - g) < \deg g.$$

„Proof” of Bartenwerfer.

The idea is to reduce the degree of f with the help of g using the proportion because „formally”

$$\deg(f - g^r) < \deg f \, .$$

Then we would replace the pair (f, g) by new one $(f - g^r, g)$ with the less degree of the first component.

The trouble is: if $r \notin \mathbb{N}$, then there may not exist the power g^r in the ring $\mathbb{C}[X, Y]$.

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The trouble is: if $r \notin \mathbb{N}$, then there may not exist the power g^r in the ring $\mathbb{C}[X, Y]$.

To overcome this problem we extend the ring $\mathbb{C}[X, Y]$ to another one in which this is possible

$$\mathbb{C}[X, Y] \subset \mathbb{C}L_\infty^{X,Y}$$

$\mathbb{C}L_\infty^{X,Y}$:={formal Laurent series at ∞ in 2 variables).

Two illustrative examples:

„Proof” of Bartenwerfer.

Example 1. Let $g(X,Y) = X^2Y^4 + X^3Y + Y + 6$. Then formally

$$g^{\frac{1}{2}}(X,Y) = \left(X^2Y^4\left(1+\frac{X}{Y^3}+\frac{1}{X^2Y^3}+\frac{6}{X^2Y^4}\right)\right)^{\frac{1}{2}}$$

$$= X^1Y^2\sqrt{1+\frac{X}{Y^3}+\frac{1}{X^2Y^3}+\frac{6}{X^2Y^4}}$$

$$= XY^2(1+\cdots)$$

We used the known formal Taylor formula $\sqrt{1+u} = 1 - ½u + \cdots$ It is well-defined because degrees of the terms in $\frac{X}{Y^3}+\frac{1}{X^2Y^3}+\frac{6}{X^2Y^4}$ are strictly negative. Then $(1+\cdots)$ is a well-defined formal series which terms are Laurent monomials in 2 variables. It is impossible in next

„Proof" of Bartenwerfer.

Example 2. Let $g(X,Y) = X^2Y^4 + X^5Y + Y + 6$. Using the same method as above

$$g^{\frac{1}{2}}(X,Y) = (X^2Y^4\left(1 + \frac{X^3}{Y^3} + \frac{1}{X^2Y^3} + \frac{6}{X^2Y^4}\right))^{½}$$

$$= X^1Y^2\sqrt{1 + \frac{X^3}{Y^3} + \frac{1}{X^2Y^3} + \frac{6}{X^2Y^4}}$$

$$= X^1Y^2(\text{???Not well-defined})$$

In formal expansion of the root we have infinite number of terms of degree 0.

„Proof” of Bartenwerfer.

$$\mathbb{C}L_{\infty}^{X,Y} := \{\sum_{i=-\infty}^{k} f_i\,, \quad f_i \in \mathbb{C}[X, Y, X^{-1}, Y^{-1}], \deg f_i = i, k \in \mathbb{Z}\}$$

- *the ring of Laurent series at* ∞ *in 2 variables*.

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One can also equivalently define

$$\mathbb{C}L_{\infty}^{X,Y} := \mathbb{C}[X,Y,\widehat{X^{-1}},Y^{-1}]$$

- the completion of the ring $\mathbb{C}[X,Y,X^{-1},Y^{-1}] = \mathbb{C}[X,Y]_{XY}$ with respect to the filtration

$$L_n := \{f \in \mathbb{C}[X,Y]_{XY} : \deg f \le -n\},\ n \in \mathbb{Z}$$

$$L_n \supset L_{n+1}, n \in \mathbb{Z}.$$

Or in other terms

$$\mathbb{C}L_{\infty}^{X,Y} := \lim_{\leftarrow} \mathbb{C}[X,Y]_{XY}/L_n$$

„Proof" of Bartenwerfer.

Examples.

$$1 + \frac{X}{Y^2} + \frac{X^2}{Y^4} + \frac{X^3}{Y^6} + \cdots \in \mathbb{C}L_\infty^{X,Y}$$

$$1 + \frac{X}{Y^2} + \frac{X^2}{Y^3} + \frac{X^3}{Y^4} + \cdots \notin \mathbb{C}L_\infty^{X,Y}$$

„Proof” of Bartenwerfer.

Properties of the ring $\mathbb{C}L_{\infty}^{X,Y}$:

1. $\mathbb{C}L_{\infty}^{X,Y}$ is a noetherian ring without zero divisors,
2. $\mathbb{C}L_{\infty}^{X,Y} \not\subset \mathbb{C}(X,Y)$, $(\sqrt{1+1/X} \notin \mathbb{C}(X,Y))$,
3. $\mathbb{C}(X,Y) \not\subset \mathbb{C}L_{\infty}^{X,Y}$, $(\frac{1}{X+Y} \notin \mathbb{C}L_{\infty}^{X,Y})$,
4. an element $f \in \mathbb{C}L_{\infty}^{X,Y}$ is invertible if and only if

$$f(X,Y) = aX^{n_1}Y^{n_2} + \downdownarrows \tag{1}$$

$\downdownarrows$ - denotes „terms of lower degree”,

5. an invertible element $f \in \mathbb{C}L_{\infty}^{X,Y}$ of form (1) has a root of dgree $k \in \mathbb{Z}^*$ if and only if $k|\, n_1$ and $k|\, n_2$,
6. $\mathbb{C}L_{\infty}^{X,Y}$ is not a local ring.

„Proof” of Bartenwerfer.

In the extension $\mathbb{C}L_{\infty}^{X,Y} \supset \mathbb{C}[X,Y]$ we modify the pair (f,g) to another one $\left(f - G(g^{1/k}), g\right)$ with the least possible degree (equal to $2-n$) of $f - G\left(g^{1/k}\right)$, where G is a Laurent polynomial in one variable and $g^{\frac{1}{k}}, k := GCD(n_1, n_2)$, is a **fixed** k-th root of g (the remaining are $\varepsilon g^{\frac{1}{k}}(X,Y)$, $\varepsilon^k = 1$).

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Properties of the modified pair:

1. $\left(f - G(g^{1/k}), g\right)$ is still a jacobian pair (in $\mathbb{C}L_{\infty}^{X,Y}$),
2. $\deg\left(f - G\left(g^{1/k}\right)\right) = 2 - n$,
3. $f - G\left(g^{1/k}\right)(X,Y) = aX^{1-n_1}Y^{1-n_2} + \Downarrow$.

„Proof” of Bartenwerfer.

After these modifications the initial pair (f, g) is reduced to another one

$$f - G\left(g^{1/k}\right)(X, Y) = aX^{1-n_1}Y^{1-n_2} + \Downarrow$$

$$g(X, Y) = X^{n_1}Y^{n_2} + \Downarrow$$

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Changing roles of f and g in the above reasoning we get yet another pair

$$f(X, Y) = X^{m_1}Y^{m_2} + \downdownarrows$$
$$g - F\left(f^{1/k}\right)(X, Y) = bX^{1-m_1}Y^{1-m_2} + \downdownarrows$$

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Moreover from $m \neq n$ and the jacobian condition it follows

$$|m_1 - m_2| \neq |n_1 - n_2|.$$

„Proof” of Bartenwerfer.

Everything up to this point is true. How does the author get a contradiction using the above reductions? Assuming that all the above formal series converge on „some set $\boldsymbol{V}$”, he calculates the cardinality of fibers of the mapping $(\boldsymbol{f},\boldsymbol{g})$ on this $\boldsymbol{V}$ in two ways:

I. Using the first reduction he calculates that this cardinality is equal to $|n_1 - n_2|$.

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II. Using the second reduction he calculates that this cardinality is equal to $|m_1 - m_2|$.

Hence he get $$|m_1 - m_2| = |n_1 - n_2|.$$

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II. Using the second reduction he calculates that this cardinality is equal to $|m_1 - m_2|$.

Hence he get $$|m_1 - m_2| = |n_1 - n_2|.$$

But we noticed above

$$|m_1 - m_2| \neq |n_1 - n_2|.$$

Contradiction. ϟϟϟ

The fault in a „proof" of Bartenwerfer.

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Steps of the author’s wrong calculations of the cardinality of fibers of (f,g):

1. Extension of the domain of the mapping (f, g).

To define appropriate set V the author extends the field $\mathbb{C}$ to another one $\mathbb{K}$ which satifies the conditions:

1. $\mathbb{C} \subset \mathbb{K}$,
2. $\mathbb{K}$ contains Laurent series in one variables at ∞.
3. $\mathbb{K}$ is algebraically closed.
4. $\mathbb{K}$ is complete with respect to some metric (non-archimedean).

This is a new point where rigid geometry is applied.

1. Extension of the domain of the mapping (f, g).

The above conditions fulfills the following construction

$$\mathbb{C} \subset \mathbb{C}[[t^{-1}]] \subset \mathbb{C}((t^{-1})) \subset \overline{\mathbb{C}((t^{-1}))} \subset \widehat{\overline{\mathbb{C}((t^{-1}))}} =: \mathbb{K}$$

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4. $\overline{\widehat{\mathbb{C}((t^{-1}))}}$ - the completion of the field $\overline{\mathbb{C}((t^{-1}))}$ in the metric

$$d(\varphi,\psi) := e^{\deg(\varphi-\psi)}.$$

1. Extension of the domain of the mapping (f, g).

Remarks:

1. $\mathbb{K}$ is algebraically closed.
2. The metric $d(\varphi,\psi)$ in $\mathbb{K}$ is non-archimedean.
3. $\mathbb{K}$ is complete.
4. In $\mathbb{K}$ we have non-archimedean absolute value

$$|\varphi| := e^{\deg(\varphi)}$$

(notice $|\varphi_n| \to 0$ iff $\deg(\varphi_n) \to -\infty$).

Definition. An absolute value of a field K (non-archimedean):

$$|\cdot|: K \to \mathbb{R}_{\geq 0}$$

1. $|a| = 0 \Leftrightarrow a = 0$,
2. $|ab| = |a||b|$,
3. $|a + b| \leq \max\{|a|, |b|\}$

2. Definition of an appropriate set V.

We define

$$V := \{(x, y) \in \mathbb{K}^2 : 1 < \varepsilon \leq |x| = |y| \leq \rho\},$$

where ε, ρ are chosen such that all the above considered series are convergent in V. This set, in this non-archimedean absolute value, is open in $\mathbb{K}^2$.

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We define

$$V := \{(x, y) \in \mathbb{K}^2 : 1 < \varepsilon \leq |x| = |y| \leq \rho\},$$

where ε, ρ are chosen such that all the above considered series are convergent in V. This set, in this non-archimedean absolute value, is open in $\mathbb{K}^2$.

Then we have the mapping

$$(f, g)_{|V} : V \to (f, g)(V).$$

We count the cardinality of fibers of this mapping.

2. Counting the cardinality of fibers of $(f,g)_{|V}$

Let $(x_0, y_0) \in V$ i

$$(f(x_0,\ y_0), g(x_0,\ y_0)) =: (\alpha, \beta).$$

From the form of f and g and properties of the absolute value $\alpha, \beta \neq 0$. We have to count $\#(f,g)_{|V}^{-1}(\alpha,\beta)$, i.e. the number of solutions in V of the system

$$f(X,Y) = \alpha$$
$$g(X,Y) = \beta$$

2. Counting the cardinality of fibers of $(f, g)_{|V}$.

Now we want to use reductions from the first part of the proof. We reduced the mapping (f, g) to

$$f - G\left(g^{1/k}\right)(X, Y) = aX^{1-n_1}Y^{1-n_2} + \Downarrow$$
$$g(X, Y) = X^{n_1}Y^{n_2} + \Downarrow$$

and

$$f(X, Y) = X^{m_1}Y^{m_2} + \Downarrow$$
$$g - F\left(f^{1/k}\right)(X, Y) = bX^{1-m_1}Y^{1-m_2} + \Downarrow$$

The above reductions were formal and used formal k-th root of the considered series (precisely fixed one roots $g^{1/k}$ and $f^{1/k}$). When we treat these series as functions in V we have to consider the remaining roots.

2. Counting the cardinality of fibers of $(f, g)_{|V}$.

To use the above reductions (precisely the first reduction) in solving in V the system

$$\begin{cases} f(X,Y) = \alpha, \\ g(X,Y) = \beta. \end{cases}$$

we should find all solutions in V of k systems

$$\begin{cases} f(X,Y) = \alpha, \\ g^{\frac{1}{k}}(X,Y) = \tilde{\beta}, \end{cases} \quad \begin{cases} f(X,Y) = \alpha, \\ \varepsilon g^{\frac{1}{k}}(X,Y) = \tilde{\beta}, \end{cases} \quad \cdots \quad \begin{cases} f(X,Y) = \alpha, \\ \varepsilon^{k-1} g^{\frac{1}{k}}(X,Y) = \tilde{\beta}, \end{cases}$$

where ε is a primitive k-th root of unity and $g^{\frac{1}{k}}(x_0,\ y_0) =: \tilde{\beta}$.
Obviously $\tilde{\beta}^k = \beta$.

2. Counting the cardinality of fibers of $(f, g)_{|V}$.

In turn each system

$$\begin{cases} f(X,Y) = \alpha, \\ \varepsilon^i g^{\frac{1}{k}}(X,Y) = \tilde{\beta}, \end{cases}$$

has the same number of solutions as the following one

$$\begin{cases} f(X,Y) - G\left(\varepsilon^i g^{\frac{1}{k}}(X,Y)\right) = \alpha - G(\tilde{\beta}), \\ \varepsilon^i g^{\frac{1}{k}}(X,Y) = \tilde{\beta}. \end{cases}$$

2. Counting the cardinality of fibers of $(f, g)_{|V}$.

The author shows that the first of these systems (for $i = 0$) i.e.

$$\begin{cases} f(X,Y) - G\left(g^{\frac{1}{k}}(X,Y)\right) = \alpha - G(\tilde{\beta}), \\ g^{\frac{1}{k}}(X,Y) = \tilde{\beta}. \end{cases}$$

has exactly $\frac{|n_1 - n_2|}{k}$ solutions in V because, by the reduction, the left hand sides have particular forms

$$f(X,Y) - G\left(g^{\frac{1}{k}}(X,Y)\right) = aX^{1-n_1}Y^{1-n_2} + \Downarrow$$

$$g^{\frac{1}{k}}(X,Y) = X^{\frac{n_1}{k}}Y^{\frac{n_2}{k}} + \Downarrow$$

2. Counting the cardinality of fibers of $(f, g)_{|V}$.

That is, the author shows that the systems for $i = 0$ i.e.

$$\begin{cases} aX^{1-n_1}Y^{1-n_2} + \downarrow\downarrow & = \alpha - G(\tilde{\beta}), \\ X^{\frac{n_1}{k}}Y^{\frac{n_2}{k}} + \downarrow\downarrow & = \tilde{\beta}. \end{cases}$$

has exactly $\frac{|n_1 - n_2|}{k}$ solutions in V. The idea of the proof will be given in a moment.

2. Author's error.

He solves only one system for $i = 0$. But we couldn't say the same on the remaining systems. In fact, for $i = 0$ the reduction gives

$$f(X,Y) - G\left(g^{\frac{1}{k}}(X,Y)\right) = aX^{1-n_1}Y^{1-n_2} + \Downarrow.$$

In the remaining cases nothing guarantees us that for all $i = 1, \ldots, k$-1, $f(X,Y) - G\left(\varepsilon^i g^{\frac{1}{k}}(X,Y)\right)$ has a similar leading form

$$f(X,Y) - G\left(\varepsilon^i g^{\frac{1}{k}}(X,Y)\right) = bX^{1-n_1}Y^{1-n_2} + \Downarrow$$

This property is crucial in the proof that the number of solutions is $\frac{|n_1 - n_2|}{k}$.

2. Author's error.

In fact, more detailed analysis leads to conclusion that there exists $i \in \{1, \dots, k-1\}$ such that the system

$$\begin{cases} f(X,Y) - G\left(\varepsilon^i g^{\frac{1}{k}}(X,Y)\right) = \alpha - G(\tilde{\beta}), \\ \varepsilon^i g^{\frac{1}{k}}(X,Y) = \tilde{\beta}. \end{cases}$$

has no solutions in V. Moreover, systems which have solutions in V, have precisely $\frac{|n_1 - n_2|}{k}$ solutions.

2. Author's error.

Therefore, from this reasoning we cannot draw the conclusion that k systems

$$\begin{cases} f(X,Y)=\alpha, \\ g^{\frac{1}{k}}(X,Y)=\tilde{\beta}, \end{cases} \quad \begin{cases} f(X,Y)=\alpha, \\ \varepsilon g^{\frac{1}{k}}(X,Y)=\tilde{\beta}, \end{cases} \quad \cdots \quad \begin{cases} f(X,Y)=\alpha, \\ \varepsilon^{k-1} g^{\frac{1}{k}}(X,Y)=\tilde{\beta}, \end{cases}$$

have

$$k \cdot \frac{|n_1-n_2|}{k} = |n_1-n_2|$$

solutions in V. In fact the above systems has less solutions in V than $|n_1-n_2|$.

This is the main author's fault.

2. Counting the cardinality of fibers of $(f, g)_{|V}$.

The idea of the proof that the system

$$f - G\left(g^{1/k}\right)(X, Y) = \alpha - G\left(\tilde{\beta}\right)$$
$$g^{1/k}(X, Y) = \tilde{\beta}$$

that is

$$\begin{cases} aX^{1-n_1}Y^{1-n_2} + \Downarrow & = \alpha - G(\tilde{\beta}), \\ X^{\frac{n_1}{k}}Y^{\frac{n_2}{k}} + \Downarrow & = \tilde{\beta}. \end{cases}$$

has exactly $\frac{|n_1 - n_2|}{k}$ solutions in V.

2. Counting the cardinality of fibers of $(f, g)_{|V}$.

First we reduce the equations to the homogeneous forms of the highest degree i.e.

$$aX^{1-n_1}Y^{1-n_2} = \alpha - G(\tilde{\beta})$$
$$X^{n_1/k}Y^{n_2/k} = \tilde{\beta}$$

This system has exactly $|n_1 - n_2|/k$ solutions in V.

2. Counting the cardinality of fibers of $(f, g)_{|V}$.

First we reduce the equations to the homogeneous forms of the highest degree i.e.

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This system has exactly $|n_1 - n_2|/k$ solutions in V.
Next we „extend" these solutions to true solutions of the initial system by using a general variant of the Hensel lemma given by Bourbaki.

2. Counting the cardinality of fibers of $(f, g)_{|V}$.

Bourbaki. Commutative Algebra, Ch. III, §4.5,Cor. 2.

Theorem. Let A be a commutative ring and $\mathfrak{m}$ an ideal in A satisfying conditions:

1. A is a topological ring with the topology given by ideals as a base of neighbourhoods of the zero,
2. A is Hausdorff and complete.
4. $\mathfrak{m}$ is closed.
5. Elements of $\mathfrak{m}$ are topologically nilpotent.

If $f = (f_1, \dots, f_n) \in A\{X_1, \dots, X_n\}^n$ and for some $a \in A^n$ the jacobian $Jac(f)(a)$ is inversible in A and $f(a) \equiv 0(mod\, \mathfrak{m})$, then there exists the unique element $x \in A^n$, such that $x \equiv a(mod\, \mathfrak{m})$ and $f(x) = 0$.

2. Counting the cardinality of fibers of $(f, g)_{|V}$.

In terms of our reasoning:

$$A = \{x \in \mathbb{K}: |x| \leq 1\} - \text{ring},$$

$$\mathfrak{m} = \{x \in \mathbb{K}: |x| < 1\} - \text{ideal},$$

$$B_i = \left\{x \in \mathbb{K}: |x| < \frac{1}{i}\right\} - \text{base}.$$

Conclusions.

I. The proof is false and couldn’t be repaired.

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II. New elements:

1. Using the Laurent series **in two variables** at infinity.
2. Using rigid geometry to cope with generalized solutions of the systems of polynomials (solutions at infinity).

Conclusions.

I. The proof is false and couldn’t be repaired.

II. New elements:

1. Using the Laurent series **in two variables** at infinity.
2. Using rigid geometry to cope with generalized solutions of the systems of polynomials (solutions at infinity).

III. The Jacobian Conjecture is still open.

Thank you for attention

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