Cent. Eur. J. Math. • 12(5) • 2014 • 688-693
DOI: 10.2478/s11533-013-0378-6

VERSITA

Central European Journal of **Mathematics**

A bound for the Milnor number of plane curve singularities

Research Article

Arkadiusz Płoski[1]*

1 *Department of Mathematics, Kielce University of Technology, Al. Tysiąclecia Państwa Polskiego 7, 25-314 Kielce, Poland*

Received 6 June 2013; accepted 24 August 2013

Abstract: Let $f = 0$ be a plane algebraic curve of degree $d > 1$ with an isolated singular point at $0 \in \mathbb{C}^2$. We show that the Milnor number $\mu_0(f)$ is less than or equal to $(d-1)^2 - [d/2]$, unless $f = 0$ is a set of d concurrent lines passing through 0, and characterize the curves $f = 0$ for which $\mu_0(f) = (d-1)^2 - [d/2]$.

MSC: 14B05, 14N99

Keywords: Milnor number • Plane algebraic curve

Introduction

Let $f \in \mathbb{C}[x, y]$ be a polynomial of degree $d > 1$ such that the curve $f = 0$ has an isolated singularity at the origin $0 \in \mathbb{C}^2$. Let $\mathcal{O} = \mathcal{O}_{\mathbb{C}^2,0}$ be the ring of germs of holomorphic functions at $0 \in \mathbb{C}^2$. The Milnor number

$$\mu_0(f) = \dim_{\mathbb{C}} \mathcal{O}\Big/\left(\tfrac{\partial f}{\partial x}, \tfrac{\partial f}{\partial y}\right)$$

is less than or equal to $(d-1)^2$ by Bézout's theorem. The equality $\mu_0(f) = (d-1)^2$ holds if and only if f is a homogeneous polynomial. The aim of this note is to determine the maximum Milnor number $\mu_0(f)$ for non-homogeneous polynomials f (Theorem 1.1) and to characterize the polynomials for which this maximum is attained (Theorem 1.4). The general problem to describe singularities that can occur on plane curves of a given degree was studied by Greuel, Lossen and Shustin in [4] (see also [9, Section 7.5] for further references). A bound for the sum of the Milnor numbers of projective hypersurfaces with isolated singular points was given recently by June Huh in [7]. Note here that a result of this type follows from Plücker–Teissier's formula for the degree of the dual hypersurface (see [8, Appendix II.3, p. 137]). To compare the two bounds let us consider the case of plane curves. Let C be a plane reduced curve of degree $d > 1$. For any

* *E-mail: matap@tu.kielce.pl*

Springer

$p \in C$ we denote $\mu_p = \mu_p(C)$ the Milnor number and $m_p = \mathrm{ord}_p C$ the order of C at p. From Plücker-Teissier's formula (see [8, Appendix II.3, p. 137] or [9, Theorem 7.2.2, p. 161]) we get $\sum_p (\mu_p + m_p - 1) \leqslant d(d-1)$.

Let o be a point of C. Then June Huh's bound (see [7, Theorem 1]) is $\sum_p \mu_p + m_o - 1 \leqslant (d-1)^2$, unless C is a cone with the apex 0.

Let us recall usual notions and conventions. By the curve $f = 0$ we mean (see [2, Chapter 3]) the linear subspace $\mathbb{C} f$ of $\mathbb{C}[x, y]$. If the polynomial f has no multiple factors then we identify the curve $f = 0$ and the set $\{P \in \mathbb{C}^2 : f(P) = 0\}$. We denote by $\mathrm{ord}_0 f$ the order of the polynomial f at $0 \in \mathbb{C}^2$ and by $i_0(f, g) = \dim_{\mathbb{C}} \mathcal{O}/_{(f,g)}$ the intersection multiplicity of the curves $f = 0$ and $g = 0$ at the origin. Then $i_0(f, g) \geqslant \mathrm{ord}_0 f \,\mathrm{ord}_0 g$ with equality if and only if the curves $f = 0$ and $g = 0$ are transverse at 0, i.e. do not have a common tangent at 0. The curve $f = 0$ has an isolated singular point at 0 if $\mathrm{ord}_0 f > 1$ and $\mu_0(f) < +\infty$. Note that f is a homogeneous polynomial of degree $d > 0$ if and only if $\mathrm{ord}_0 f = d$.

1. Results

For any $a \in \mathbb{R}$ we denote by $[a]$ the integer part of a. The main result of this note is

Theorem 1.1.
Let $f = 0$ be a curve of degree $d > 1$ with an isolated singular point at $0 \in \mathbb{C}^2$. Suppose that $\mathrm{ord}_0 f < d$. Then

$$\mu_0(f) \leqslant (d-1)^2 - \left[\frac{d}{2}\right].$$

We prove Theorem 1.1 in Section 3. The bound in the theorem is exact.

Example 1.2.
Let $d > 1$ be an integer. Put

$$f(x, y) = \begin{cases} \displaystyle\prod_{k=1}^{d/2} \left(x + kx^2 + y^2\right) & \text{if} \quad d \equiv 0 \pmod 2, \\ \displaystyle x \prod_{k=1}^{(d-1)/2} \left(x + kx^2 + y^2\right) & \text{if} \quad d \not\equiv 0 \pmod 2. \end{cases}$$

Then f is a polynomial of degree d and $\mu_0(f) = (d-1)^2 - [d/2]$.

Remark 1.3.
Gusein-Zade and Nekhoreshev using topological methods proved in [5, Proposition 1] that if $\mathrm{ord}_0 f = 2$ then $\mu_0(f) \leqslant (d-1)^2 - [d/2]([d/2] - 1)$. Another bound for the Milnor number follows from the Abhyankar–Moh theory of approximate roots (see [3, Corollary 6.5]). Suppose that the curve $f = 0$ is unibranch at 0 (i.e. f is irreducible in the ring of formal power series $\mathbb{C}[[x, y]]$) and the unique tangent to $f = 0$ at 0 intersects the curve with multiplicity d. Then $\mu_0(f) \leqslant (d-1)^2 - (d/d_1 - 1)(d - \mathrm{ord}_0 f)$, where $d_1 = \gcd(\mathrm{ord}_0 f, d)$.

Theorem 1.4.
Let f be a polynomial of degree $d > 2$, $d \neq 4$. Then the following two conditions are equivalent:

(i) *The curve $f = 0$ passes through the origin and $\mu_0(f) = (d-1)^2 - [d/2]$,*

(ii) *The curve $f = 0$ has $d - [d/2]$ irreducible components. Each irreducible component of the curve passes through the origin. If $d \equiv 0 \pmod 2$ then all components are of degree 2 and intersect pairwise at 0 with multiplicity 4. If $d \not\equiv 0 \pmod 2$ then all but one component are of degree 2 and intersect pairwise at 0 with multiplicity 4, the remaining component is linear and is tangent to all components of degree 2.*

The proof of Theorem 1.4 is given in Section 4.

Remark 1.5.

The implication (ii) $\Rightarrow$ (i) holds for any $d > 2$. The assumption $d \neq 4$ is necessary for the implication (i) $\Rightarrow$ (ii). Take $f(x,y) = x\big(y^3 - x^2\big)$ which is a simple singularity of type E_7. Then f is of degree $d = 4$, $\mu_0(f) = (d-1)^2 - [d/2] = 7$ and condition (ii) fails.

2. Preparatory lemmas

Let us begin with the following well-known properties of the Milnor number.

Lemma 2.1.

(i) *If $f = f_0\widetilde{f}$ in $\mathbb{C}[x,y]$ with $f_0(0) \neq 0$ then $\mu_0(f) = \mu_0(\widetilde{f})$.*

(ii) *If $f = f_1 \cdots f_m$ in $\mathbb{C}[x,y]$, $f_i(0) = 0$ for $i = 1, \ldots, m$, then*

$$\mu_0(f) + m - 1 = \sum_{i=1}^{m} \mu_0(f_i) + 2 \sum_{1\leqslant i<j\leqslant m} i_0(f_i, f_j).$$

For a proof where the topological arguments are used see [9, Theorem 6.5.1, p. 145]. For a proof based on the properties of intersection numbers see e.g. [1, Proposition 2.4, p. 13].

Lemma 2.2.

Let f be an irreducible polynomial of degree $d > 1$, $f(0) = 0$. Then $\mu_0(f) \leqslant (d-1)(d-2)$ with equality if and only if the curve $f = 0$ is rational, its projective closure C has exactly one singular point $0 \in \mathbb{C}^2$ and $f = 0$ is unibranch at 0.

Proof. Apply the formula for the genus g of C (see [9, Corollary 7.1.3]): $2g = (d-1)(d-2) - \sum_P (\mu_P + r_P - 1)$, where r_P is the number of branches of C passing through P. $\square$

Example 2.3.

The polynomial $f = x^{d-1} + y^d$ is irreducible and $\mu_0(f) = (d-1)(d-2)$.

Let $f = 0$ be a curve of degree $d > 1$ with an isolated singular point at $0 \in \mathbb{C}^2$. Let $f_i = 0$, $i = 1, \ldots, m$, be irreducible components of $f = 0$ passing through 0 and let $d_i = \deg f_i$ for $i = 1, \ldots, m$. Then $f = f_0 f_1 \cdots f_m$ in $\mathbb{C}[x,y]$, where $f_0(0) \neq 0$.

Lemma 2.4.

Let $\Lambda = \big\{(i,j) \in \mathbb{N}^2 : 1 \leqslant i < j \leqslant m,\ f_i = 0,\ f_j = 0$ are transverse (i.e. their tangent cones intersect only at 0) and $d_i > 1$ or $d_j > 1\big\}$. Then $\mu_0(f) \leqslant (d-1)^2 - d + m - 2 \cdot \#\Lambda$.

Proof. Let $\widetilde{f} = f_1 \cdots f_m$. Observe that for $(i,j) \in \Lambda$ we have $i_0(f_i, f_j) = (\mathrm{ord}_0 f_i)(\mathrm{ord}_0 f_j) < d_i d_j$ since $d_i > 1$ or $d_j > 1$ (if f_i is irreducible of degree $d_i > 1$ then $\mathrm{ord}_0 f_i < d_i$). By Lemma 2.2 we get $\mu_0(f_i) \leqslant (d_i - 1)(d_i - 2)$ for $i = 1, \ldots, m$. Now, Lemma 2.1 implies

$$\begin{aligned}\mu_0(f) + m - 1 = \mu_0(\widetilde{f}) + m - 1 &\leqslant \sum_{i=1}^{m} (d_i - 1)(d_i - 2) + 2\sum_{(i,j)\notin\Lambda} i_0(f_i, f_j) + 2\sum_{(i,j)\in\Lambda} i_0(f_i, f_j)\\ &\leqslant \sum_{i=1}^{m} (d_i - 1)(d_i - 2) + 2\sum_{(i,j)\notin\Lambda} d_i d_j + 2\sum_{(i,j)\in\Lambda} (d_i d_j - 1) = (d-1)^2 - d + 2m - 2 \cdot \#\Lambda - 1 \qquad \square\end{aligned}$$

Lemma 2.5.
We have that $\mu_0(f) \leqslant (d-1)^2 - d + m$. The equality $\mu_0(f) = (d-1)^2 - d + m$ holds if and only if $\mu_0(f_i) = (d_i - 1)(d_i - 2)$ and $i_0(f_i, f_j) = d_i d_j$ for $1 \leqslant i < j \leqslant m$.

Proof. The inequality $\mu_0(f) \leqslant (d-1)^2 - d + m$ follows from Lemma 2.4 (see also [6, Proposition 6.3]). By Lemma 2.1 we can rewrite the equality $\mu_0(f) = (d-1)^2 - d + m$ in the form

$$\sum_{i=1}^{m} \mu_0(f_i) + 2 \sum_{1 \leqslant i < j \leqslant m} i_0(f_i, f_j) = \sum_{i=1}^{m} (d_i - 1)(d_i - 2) + 2 \sum_{1 \leqslant i < j \leqslant m} d_i d_j.$$

We have that $\mu_0(f_i) \leqslant (d_i - 1)(d_i - 2)$ by Lemma 2.2 and $i_0(f_i, f_j) \leqslant d_i d_j$ by Bézout's theorem which together with the equality above imply the lemma. □

Lemma 2.6.
Let $d > 2$. If $\mu_0(f) = (d-1)^2 - d + m$ then all irreducible components of the curve $f = 0$ pass through $0 \in \mathbb{C}^2$.

Proof. Let $\widetilde{d} = \deg \widetilde{f}$. Clearly $\widetilde{d} \leqslant d$. We have $(d-1)^2 - d + m = \mu_0(f) = \mu_0(\widetilde{f})$ and $\mu_0(\widetilde{f}) \leqslant (\widetilde{d} - 1)^2 - \widetilde{d} + m$ by Lemma 2.5. The inequalities $d > 2$, $\widetilde{d} \leqslant d$ and $(d-1)^2 - d \leqslant (\widetilde{d} - 1)^2 - \widetilde{d}$ imply $\widetilde{d} = d$. Therefore $\deg f_0 = d - \widetilde{d} = 0$ and f_0 is a constant. □

Lemma 2.7.
$\#\{i \in [1, m] : d_i > 1\} \leqslant d - m$.

Proof. Let $I = \{i \in [1, m] : d_i > 1\}$. Then $I = \{i \in [1, m] : \mathrm{ord}_0 f_i < d_i\}$ and $\# I \leqslant \sum_{i \in I} (d_i - \mathrm{ord}_0 f_i) = \sum_{i=1}^{m} (d_i - \mathrm{ord}_0 f_i) \leqslant d - \mathrm{ord}_0 f \leqslant d - m$. □

A line $l = 0$ is tangent to the curve $f = 0$ (at $0 \in \mathbb{C}^2$) if $i_0(f, l) > \mathrm{ord}_0 f$. We denote by $T(f)$ the set of all tangents at 0 to $f = 0$. We have $\# T(f) \leqslant \mathrm{ord}_0 f \leqslant \deg f - 1$ if f is not homogeneous. For two polynomials f, g, $T(fg) = T(f) \cup T(g)$. Therefore we get $\# T(fg) \leqslant \# T(f) + \# T(g) - 1$ if $T(f) \cap T(g) \neq \emptyset$.

Lemma 2.8.
Let f_i, $i = 1, \dots, k$, be irreducible polynomials of degree $d_i > 1$ such that $f_i(0) = 0$ for $i = 1, \dots, k$. Suppose that the curves $f_i = 0$, $i = 1, \dots, k$, have a common tangent at 0. Then $\# T(f_1 \cdots f_k) \leqslant \sum_{i=1}^{k} (d_i - 1) - k + 1$.

Proof. If $k = 1$ it is clear. Suppose that $k > 1$ and that the lemma is true for the sequences of $k - 1$ polynomials. Let $f_1, \dots, f_k$ be a sequence of k irreducible polynomials of degree > 1 such that the curves $f_i = 0$, $i = 1, \dots, k$, have a common tangent at 0. Then by the induction hypothesis $\# T(f_1 \cdots f_{k-1}) \leqslant \sum_{i=1}^{k-1} (d_i - 1) - (k - 1) + 1$. On the other hand $\# T(f_k) \leqslant d_k - 1$ and we get $\# T(f_1 \cdots f_k) \leqslant \# T(f_1 \cdots f_{k-1}) + \# T(f_k) - 1 \leqslant \sum_{i=1}^{k} (d_i - 1) - (k - 1)$, since $f_1 \cdots f_{k-1}$ and f_k have a common tangent. □

3. Proof of Theorem 1.1

Let f be a polynomial of degree $d > 1$ such that $f(0) = 0$ and $\mu_0(f) < +\infty$. We assume that f is not homogeneous. Let m be the number of irreducible components of the curve $f = 0$ passing through $0 \in \mathbb{C}^2$.

Lemma 3.1.
If $m \leqslant d - [d/2]$ then $\mu_0(f) \leqslant (d-1)^2 - [d/2]$. The equality $\mu_0(f) = (d-1)^2 - [d/2]$ implies $m = d - [d/2]$.

Proof. Suppose that $m \leqslant d-[d/2]$. By the first part of Lemma 2.5 we get $\mu_0(f) \leqslant (d-1)^2-d+m \leqslant (d-1)^2-[d/2]$. If $\mu_0(f) = (d-1)^2-[d/2]$ then $(d-1)^2-d+m = (d-1)^2-[d/2]$, so $m = d-[d/2]$. $\square$

Lemma 3.2.
If $m \geqslant d-[d/2]+1$ then $\mu_0(f) < (d-1)^2-[d/2]$.

Proof. Write $f = f_0 f_1 \cdots f_m$, where $f_0(0) \neq 0$ and f_i are irreducible with $f_i(0) = 0$ for $i = 1, \dots, m$. Since the sequence $d \mapsto d-[d/2]$ is increasing it suffices to check the lemma for the polynomial $\widetilde{f} = f_1 \cdots f_m$. In what follows we write f instead of $\widetilde{f}$ and put $d_i = \deg f_i$ for $i = 1, \dots, m$. Since f is not homogeneous we have $m < d$. Let $k = \#\{i : d_i > 1\}$. From $d_1 + \dots + d_m = d$ it follows that $k \geqslant 1$. Note also that $m-k = \#\{i : d_i = 1\} > 0$ since by Lemma 2.7 we have $k \leqslant d-m$ and consequently $m-k \geqslant m-(d-m) = 2m-d \geqslant d-2[d/2]+2 > 0$. We label $f_1, \dots, f_m$ so $d_1 \geqslant \dots \geqslant d_m$. Therefore we get $d_1 \geqslant \dots \geqslant d_k \geqslant 2$ and $d_{k+1} = \dots = d_m = 1$. Let us consider two cases.

Case 1. The curves $f_1 = 0, \dots, f_k = 0$ have a common tangent. By Lemma 2.8 we have $\# T(f_1 \cdots f_k) \leqslant \sum_{i=1}^{k}(d_i-1) - k+1 = \sum_{i=1}^{m}(d_i-1)-k+1 = d-m-k+1$. Therefore we get $m-k-\# T(f_1 \cdots f_k) \geqslant m-k-(d-m-k+1) = 2m-d-1$. Note that $2m-d-1 \geqslant 2(d-[d/2]+1)-d+1 > 0$. Thus there are at least $2m-d-1 > 0$ linear forms in the sequence $f_{k+1}, \dots, f_m$ that are transverse to the curve $f_1 \cdots f_k = 0$ and we get $\#\Lambda \geqslant k(2m-d-1) \geqslant 2m-d-1$. Consequently by Lemma 2.4 we obtain $\mu_0(f) \leqslant (d-1)^2-d+m-2(2m-d-1) = (d-1)^2+d-3m+2 \leqslant (d-1)^2+d-3(d-[d/2]+1)+2 = (d-1)^2-[d/2]-2(d-[d/2])-1 < (d-1)^2-[d/2]$.

Case 2. The curves $f_1 = 0, \dots, f_k = 0$ have no common tangent. Then for every linear form f_j, $k+1 \leqslant j \leqslant m$, there exists a polynomial f_i, $1 \leqslant i \leqslant k$, such that f_i, f_j are transverse. Therefore $\#\Lambda \geqslant m-k$ and by Lemma 2.4 we get $\mu_0(f) \leqslant (d-1)^2-d+m-2(m-k) = (d-1)^2-d-m+2k$. Since by Lemma 2.7 we have $k \leqslant d-m$ the above bound for $\mu_0(f)$ implies $\mu_0(f) \leqslant (d-1)^2+d-3m \leqslant (d-1)^2+d-3(d-[d/2]+1) = (d-1)^2-[d/2]-2(d-2[d/2])-3 < (d-1)^2-[d/2]$. $\square$

Now from Lemmas 3.1 and 3.2 we get $\mu_0(f) \leqslant (d-1)^2-[d/2]$ which proves Theorem 1.1.

4. Proof of Theorem 1.4

To show that a curve $f = 0$ of degree $d > 4$ with $\mu_0(f) = (d-1)^2-[d/2]$ does not have irreducible components of degree 3 we need

Lemma 4.1.
Let $f, g \in \mathbb{C}[x,y]$ be irreducible polynomials, $\deg f = 3$, $\deg g = 2$, $f(0) = g(0) = 0$. Suppose that the curve $f = 0$ has a singular point at 0 and $\# T(f) = 1$. Then

$$i_0(f,g) < \deg f \deg g = 6.$$

Proof. If $f = 0$ and $g = 0$ have no common tangent then $i_0(f,g) = \mathrm{ord}_0 f\, \mathrm{ord}_0 g = \mathrm{ord}_0 f < 3$. Thus we may assume that $f = x^2 + f^+$, $g = x + g^+$, where f^+, g^+ are homogeneous forms of degree 3 and 2, respectively. We get $i_0(f,g) = i_0(f-xg, g) = i_0(f^+-xg^+, g) = 3 \cdot 1$ for $f^+-xg^+ = 0$ and $g = 0$ have no common tangent. $\square$

Lemma 4.2.
Let f be a polynomial of degree $d > 2$ such that $f(0) = 0$. Suppose that $\mu_0(f) = (d-1)^2-d+m$, where m is the number of irreducible components of the curve $f = 0$ passing through $0 \in \mathbb{C}^2$. Then $f = f_1 \cdots f_m$ in $\mathbb{C}[x,y]$ with irreducible f_i, $f_i(0) = 0$ for $i = 1, \dots, m$. Let $d_i = \deg f_i$ for $i = 1, \dots, m$. Then $\# T(f_i) = 1$ and $i_0(f_i, f_j) = d_i d_j$ for $i < j$. If $m < d$ then $f = 0$ has at most one linear component and has no two components of degree 2 and 3.

Proof. By Lemmas 2.5 and 2.6 we get $f = f_1 \cdots f_m$ in $\mathbb{C}[x,y]$, f_i irreducible, $f_i(0) = 0$, $i_0(f_i, f_j) = d_i d_j$ and $\mu_0(f_i) = (d_i-1)(d_i-2)$. We have $\# T(f_i) = 1$ for the curve $f_i = 0$ has only one branch at 0 by Lemma 2.2.

Suppose that the curve $f = 0$ has two linear components $f_j = 0$ and $f_k = 0$, $j \neq k$. Then there is no component $f_i = 0$ of $f = 0$ of degree $d_i > 1$ (if $f_i = 0$ had degree $d_i > 1$ then we would get $i_0(f_i, f_j) = i_0(f_i, f_k) = d_i > 1$ which is impossible for $\#T(f_i) = 1$). Therefore if $f = 0$ has two linear components then all components are linear and intersect pairwise with multiplicity 1. Thus $m = d$ and f is a homogeneous form of degree d.

Therefore, if $m < d$ then there exists at most one linear component. Since $i_0(f_i, f_j) = d_i d_j$ if $i < j$ there are no two components of degree 2 and 3 by Lemma 4.1. □

Now, we can pass to the proof of Theorem 1.4.

Proof of Theorem 1.4. (i) ⇒ (ii) Assume that $\mu_0(f) = (d-1)^2 - [d/2]$, where $d > 1$ and $d \not\equiv 4$. Then $f = 0$ has $m = d - [d/2]$ irreducible components passing through $0 \in \mathbb{C}^2$ by Lemmas 3.1 and 3.2. We have $\mu_0(f) = (d-1)^2 - [d/2] = (d-1)^2 - d + m$ with $m = d - [d/2] < d$ and by Lemma 4.2 we can write $f = f_1 \cdots f_m$, $f_i \in \mathbb{C}[x, y]$ irreducible, $f_i(0) = 0$. We label $f_1, \dots, f_m$ so that $d_1 \geqslant \dots \geqslant d_m \geqslant 1$.

Case 1: $d \equiv 0 \pmod 2$. Then $m = d/2$ and $d_1 + \cdots + d_m = d$. This is possible if and only if $(d_1, \dots, d_m) = (2, \dots, 2)$ or $(d_1, \dots, d_m) = (3, 2, \dots, 2, 1)$, where 2 appears $m - 2 = d/2 - 2 > 0$ times since $d > 4$. If $(d_1, \dots, d_m) = (2, \dots, 2)$ then the theorem follows from Lemma 4.2. The case $(d_1, \dots, d_m) = (3, 2, \dots, 2, 1)$ cannot occur by Lemma 4.1.

Case 2. Case 2: $d \not\equiv 0 \pmod 2$. In this case we have $m = (d+1)/2$. From $d_1 + \cdots + d_m = d$ it follows that $(d_1, \dots, d_m) = (2, \dots, 2, 1)$. We apply Lemma 4.2.

The implication (ii) ⇒ (i) follows immediately from Lemma 2.1 (ii). □

References

[1] Cassou-Noguès P., Płoski A., Invariants of plane curve singularities and Newton diagrams, Uni. Iagel. Acta Math., 2011, 49, 9–34

[2] Fulton W., Algebraic Curves, Adv. Book Classics, Addison-Wesley, Redwood City, 1989

[3] García Barroso E.R., Płoski A., An approach to plane algebroid branches, preprint available at http://arxiv.org/abs/1208.0913

[4] Greuel G.-M., Lossen C., Shustin E., Plane curves of minimal degree with prescribed singularities, Invent. Math., 1998, 133(3), 539–580

[5] Gusein-Zade S.M., Nekhoroshev N.N., Singularities of type A_k on plane curves of a chosen degree, Funct. Anal. Appl., 2000, 34(3), 214–215

[6] Gwoździewicz J., Płoski A., Formulae for the singularities at infinity of plane algebraic curves, Univ. Iagel. Acta Math., 2001, 39, 109–133

[7] Huh J., Milnor numbers of projective hypersurfaces with isolated singularities, preprint available at http://arxiv.org/abs/1210.2690

[8] Teissier B., Resolution Simultanée I, II, In: Séminaire sur les Singularités des Surfaces, Lecture Notes in Math., 777, Springer, Berlin, 1980, 71–146

[9] Wall C.T.C., Singular Points of Plane Curves, London Math. Soc. Stud. Texts, 63, Cambridge University Press, Cambridge, 2004