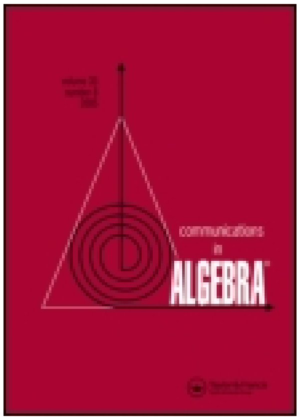

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A NOTE ON THE DISCRIMINANT

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Let $F(X, Y) = Y^d + a_1(X)Y^{d-1} + \cdots + a_d(X)$ be a polynomial in $n+1$ variables $(X, Y) = (X_1, \ldots, X_n, Y)$ with coefficients in an algebraically closed field $\mathbb{K}$. Assuming that the discriminant $D(X) = \mathrm{disc}_Y F(X, Y)$ is nonzero we investigate the order $\mathrm{ord}_P D$ for $P \in \mathbb{K}^n$. As application we get a discriminant criterion for the hypersurface $F = 0$ to be nonsingular.

Key Words: Discriminant of a polynomial; Nonsingular hypersurface.

2010 Mathematics Subject Classification: 11C08; 13B25.

PRELIMINARIES

Let $\mathbb{K}$ be a fixed algebraically closed field of arbitrary characteristic. A nonconstant polynomial F in $N > 0$ variables defines a hypersurface $F = 0$ which is by definition the set of all polynomials aF where $a \in \mathbb{K}\backslash\{0\}$. Let $V(F) = \{P \in \mathbb{K}^N : F(P) = 0\}$ be the set of zeroes of the polynomial F. The order $\mathrm{ord}_P F$ is the lowest degree in the expansion of F at P obtained by translation of coordinates. Thus $\mathrm{ord}_P F > 0$ if and only if $P \in V(F)$. A point $P \in \mathbb{K}^N$ is a simple (or nonsingular) point of the hypersurface $F = 0$ if $\mathrm{ord}_P F = 1$. If all points $P \in V(F)$ are simple the hypersurface $F = 0$ is called nonsingular.

For the notion of discriminant which is basic in this article we refer the reader to Abhyankar's book [1], Lecture L4 (see also appendix IV of [4] or [2, § 42]).

1. RESULT

Let $F(X, Y) = Y^d + a_1(X)Y^{d-1} + \cdots + a_d(X) \in \mathbb{K}[X, Y]$ be a polynomial in $n+1$ variables $(X, Y) = (X_1, \ldots, X_n, Y)$ of degree $d > 0$ in Y. Let $D(X) = \mathrm{disc}_Y F(X, Y)$ be the Y-discriminant of F (if $d = 1$, then $D(X) = 1$) and assume that $D(X) \neq 0$ in $\mathbb{K}[X]$. Then F has no multiple factors in $\mathbb{K}[X, Y]$. Let $\pi : V(F) \to \mathbb{K}^n$ be the projection given by $\pi(a_1, \ldots, a_n, b) = (a_1, \ldots, a_n)$. In the sequel, we write $P = (a_1, \ldots, a_n)$ and $Q = (a_1, \ldots, a_n, b)$. We put $\mathrm{mult}_Q \pi = \mathrm{ord}_b F(P, Y)$ for $Q \in V(F)$.

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Then

$$\sum_{Q\in\pi^{-1}(P)} \mathrm{mult}_Q\pi = d$$

for $P \in \mathbb{K}^n$. Obviously, we have $\#\pi^{-1}(P) \le d$ with equality if and only if $P \notin V(D)$ (see [1], Lecture L4, Observation (0.2)).

The main result of this note is the following theorem.

Theorem 1.1. *For any $P \in \mathbb{K}^n$ we have $\mathrm{ord}_P D \ge d - \#\pi^{-1}(P)$. The equality $\mathrm{ord}_P D = d - \#\pi^{-1}(P)$ holds if and only if the following two conditions are satisfied*:

(i) $\mathrm{ord}_Q F = 1$ *for all $Q \in \pi^{-1}(P)$ i.e., all points of $\pi^{-1}(P)$ are nonsingular*;
(ii) *If* $\mathrm{char}\mathbb{K} = p$ *then p does not divide the multiplicities* $\mathrm{mult}_Q\pi$ *for all $Q \in \pi^{-1}(P)$.*

We prove Theorem 1.1 in Section of this note. Observe that if $\mathrm{char}\,\mathbb{K} = 0$, then only Condition (i) is relevant. Let us note a few corollaries to Theorem 1.1.

Corollary 1.2. *If* $\mathrm{ord}_P D = 0$ *or* $\mathrm{ord}_P D = 1$, *then all points of $\pi^{-1}(P)$ are nonsingular.*

Proof. If $\mathrm{ord}_P D = 0$, then $P \notin V(D)$ and $\#\pi^{-1}(P) = d$ that is $\mathrm{ord}_P D = d - \#\pi^{-1}(P) = 0$ and all points of $\pi^{-1}(P)$ are nonsingular by Theorem 1.1(i). If $\mathrm{ord}_P D = 1$, then by the first part of Theorem 1.1, we have $\#\pi^{-1}(P) \ge d - 1$. In fact, $\#\pi^{-1}(P) = d - 1$ because if $\#\pi^{-1}(P) = d$, then $P \notin V(D)$ and $\mathrm{ord}_P D = 0$. We have then $\mathrm{ord}_P D = d - \#\pi^{-1}(P) = 1$ and again by Theorem 1.1(i), we get the assertion. □

Corollary 1.3. *If Q is a singular point of $F = 0$ (i.e.,* $\mathrm{ord}_Q F > 1$), *then* $\mathrm{mult}_Q\pi \le \mathrm{ord}_{\pi(Q)} D$.

Proof. Let $P = \pi(Q)$. Since $Q \in \pi^{-1}(P)$ is singular, we get by Theorem 1.1 that $\mathrm{ord}_P D > d - \#\pi^{-1}(P)$. On the other hand, it is easy to check that $\mathrm{mult}_Q\pi \le d - \#\pi^{-1}(P) + 1$. Therefore, we have $\mathrm{mult}_Q\pi \le \mathrm{ord}_{\pi(Q)} D$. □

The following property of the discriminant is well-known in the case $\mathbb{K} = \mathbf{C}$ (see [4, Appendix IV, Theorem 11 B]).

Corollary 1.4. *Let $\mathbf{F}(A, Y) = Y^d + A_1 Y^{d-1} + \cdots + A_d \in \mathbf{Z}[A, Y]$ be the polynomial with undeterminate coefficients $A = (A_1, \ldots, A_d)$ and $\mathbf{D}(A) = \mathrm{disc}_Y \mathbf{F}(A, Y)$. Let $r(a)$ be the number of distinct roots of the polynomial $\mathbf{F}(a, Y) \in \mathbb{K}[Y]$ where $a = (a_1, \ldots, a_d) \in \mathbb{K}^d$. Then* $\mathrm{ord}_a \mathbf{D}(A) \ge d - r(a)$. *The equality* $\mathrm{ord}_a \mathbf{D}(A) = d - r(a)$ *holds if* $\mathrm{char}\,\mathbb{K} = 0$ *or* $\mathrm{char}\,\mathbb{K} = p$ *and p does not divide the multiplicities of roots of the polynomial* $\mathbf{F}(a, Y)$.

Proof. Observe that the hypersurface $\mathbf{F}(A, Y) = 0$ is nonsingular and use Theorem 1.1.

Corollary 1.5 (Discriminant Criterion for Nonsingular Hypersurfaces). *Assume that $\mathbb{K}$ is of characteristic zero. Then the hypersurface defined by*

$$F(X, Y) = Y^d + a_1(X)Y^{d-1} + \cdots + a_d(X) = 0$$

is nonsingular if and only if

$$\operatorname{ord}_P \mathbf{D}(a_1(X), \dots, a_d(X)) = \operatorname{ord}_{a(P)} \mathbf{D}(A_1, \dots, A_d),$$

where $a(P) = (a_1(P), \dots, a_d(P))$, *for all singular points* P *of the discriminant hypersurface* $D(X) = 0$.

Proof. Using Theorem 1.1 and Corollary 1.2, we see that the hypersurface $F(X, Y) = 0$ is nonsingular if and only if $\operatorname{ord}_P D(X) = d - \#\pi^{-1}(P)$ for all $P \in \mathbb{K}^n$ such that $\operatorname{ord}_P D(X) > 1$. On the other hand, by Corollary 1.4, we have that $d - \#\pi^{-1}(P) = \operatorname{ord}_{a(P)} \mathbf{D}(A_1, \dots, A_d)$. By definition of the discriminant, $D(X) = \mathbf{D}(a_1(X), \dots, a_d(X))$ and the corollary follows. □

2. A PROPERTY OF THE DISCRIMINANT

In the proof of Theorem 1.1, we need the following property of the discriminant.

Lemma 2.1. *Let* $\mathbf{D}(A_1, \dots, A_d)$ *be the discriminant of the general polynomial* $\mathbf{F} = Y^d + A_1 Y^{d-1} + \cdots + A_d$. *Then*

$$\mathbf{D}(A_1, \dots, A_d) = (-1)^{\binom{d}{2}} d^d A_d^{d-1} + \alpha_1(A_1, \dots, A_{d-1}) A_d^{d-2} + \cdots + \alpha_{d-1}(A_1, \dots, A_{d-1})$$

in $\mathbf{Z}[A_1, \dots, A_d]$, *where* $\operatorname{ord}_0 \alpha_i(A_1, \dots, A_{d-1}) \geq i + 1$ *for* $i = 1, \dots, d-1$.

Proof. Let

$$I = \left\{ (p_1, \dots, p_d) \in \mathbf{Z}^d : \ p_i \geq 0 \text{ for } i = 1, \dots, d \text{ and } \sum_{i=1}^{d} i p_i = d(d-1) \right\}.$$

Then the discriminant $\mathbf{D}(A_1, \dots, A_d)$ is equal to the sum of monomials of the form $c_{p_1, \dots, p_d} A_1^{p_1} \dots A_d^{p_d}$ where $(p_1, \dots, p_d) \in I$ (see [1], Lecture L.4, Observation (03) or [2],§ 42). It is easy to see that:

(a) If $(p_1, \dots, p_d) \in I$, then $p_d \leq d - 1$;
(b) If $(p_1, \dots, p_d) \in I$, then $\sum_{i=1}^{d} p_i \geq d - 1$ with equality if and only if $(p_1, \dots, p_d) = (0, \dots, 0, d-1)$.

Therefore, we get

$$\mathbf{D}(A_1, \dots, A_d) = c_{0, \dots, 0, d} A_d^{d-1} + \alpha_1(A_1, \dots, A_{d-1}) A_d^{d-2} + \cdots + \alpha_{d-1}(A_1, \dots, A_{d-1}),$$

where $\operatorname{ord}_0 \alpha_i(A_1, \dots, A_{d-1}) > i$ for $i = 1, \dots, d-1$ and

$$c_{0, \dots, 0, d} = \mathbf{D}(0, \dots, 0, 1) = \operatorname{disc}_Y(Y^d + 1) = (-1)^{\binom{d}{2}} d^d$$ □

Remark 2.2. Lemma 2.1 implies that $\operatorname{ord}_0 D(A_1, \dots, A_d) = d - 1$. Let $d(A_1, \dots, A_{d-1}) = \operatorname{disc}_Y(Y^{d-1} + A_1 Y^{d-2} + \cdots + A_{d-1})$. Then $\operatorname{ord}_0 d(A_1, \dots, A_{d-1}) = d - 2$. It is easy to check that $\mathbf{D}(A_1, \dots, A_{d-1}, 0) = \mathrm{disc}_Y(Y^d + A_1 Y^{d-1} + \cdots + A_{d-1}Y) = d(A_1, \dots, A_{d-1})A_{d-1}^2$. Therefore, $\mathrm{ord}_0 \alpha_{d-1}(A_1, \dots, A_{d-1}) = \mathrm{ord}_0 \mathbf{D}(A_1, \dots, A_{d-1}, 0) = \mathrm{ord}_0 d(A_1, \dots, A_{d-1}) + 2 = (d-2) + 2 = d$.

From Lemma 2.1 and Remark 2.2 it follows that the Newton diagram $\{(\mathrm{ord}_0 \alpha_k, d-k-1) : \alpha_k \neq 0\}$ of the polynomial $\mathbf{D} = \alpha_0 A_d^{d-1} + \alpha_1 A_d^{d-2} + \cdots + \alpha_{d-1} \in \mathbf{Z}[A_1, \dots, A_{d-1}][A_d]$ intersects the axes at points $(0, d-1)$ and $(d, 0)$. All remaining points of the diagram lie strictly above the segment joining these two points.

3. PROOF

Let $f(X, Y) \in \mathbb{K}[[X, Y]]$ be a formal power series distinguished in Y with order $d > 0$, i.e., such that $\mathrm{ord}\, f(0, Y) = d$. Then by the Weierstrass Preparation Theorem $f(X, Y) = F(X, Y)U(X, Y)$ in $\mathbb{K}[[X, Y]]$ where $F(X, Y) = Y^d + a_1(X)Y^{d-1} + \cdots + a_d(X) \in \mathbb{K}[[X]][Y]$ is a distinguished polynomial and $U(0, 0) \neq 0$.

Let $D(X) = \mathrm{disc}_Y F(X, Y)$. Then we define $\tilde{\mu}(f) = \mathrm{ord}_0 D(X) - d + 1$ if $D(X) \neq 0$ and $\tilde{\mu}(f) = +\infty$ if $D(X) = 0$.

Lemma 3.1. *We have $\tilde{\mu}(f) \geq 0$. The equality $\tilde{\mu}(f) = 0$ holds if and only if*:

(i) $\mathrm{ord}_0 f = 1$;
(ii) *If* $\mathrm{char}\, \mathbb{K} = p$, *then p does not divide d.*

Proof. Let $D(X) \neq 0$. By Lemma 2.1, we have $\mathrm{ord}_0 \mathbf{D}(A_1, \dots, A_d) = d - 1$ so $\mathrm{ord}_0 \mathbf{D}(X) = \mathrm{ord}_0 \mathbf{D}(a_1(X), \dots, a_d(X)) \geq d - 1$ since $\mathrm{ord}_0 a_i(X) \geq 1$ for $i = 1, \dots, d$. Again by Lemma 2.1, we can write

$$D(X) = (-1)^{\binom{d}{2}} d^d a_d(X)^{d-1} + \widetilde{D}(X),$$

where $\mathrm{ord}_0 \widetilde{D}(X) > d - 1$. Thus if $\mathrm{ord}_0 a_d(X) = 1$ and p does not divide d, then $\mathrm{ord}_0 D(X) = d - 1$. If $\mathrm{ord}_0 a_d(X) > 1$ or p divides d, then $\mathrm{ord}_0 D(X) > d - 1$. This proves the lemma since $\mathrm{ord}_0 F(X, Y) = 1$ if and only if $\mathrm{ord}_0 a_d(X) = 1$, and $\mathrm{ord}_0 f = \mathrm{ord}_0 F$. □

Remark 3.2 (see [3, Lemma 5.10]). If $\mathrm{char}\, \mathbb{K} = 0$, then $\tilde{\mu}(f)$ is equal to the Milnor number of the algebroid curve $f(cT, Y) = 0$, where $X_i = c_i T$ $(i = 1, \dots, r)$ is a line intersecting transversally the discriminant hypersurface $D(X) = 0$.

Lemma 3.3. *Let* $F(X, Y) = Y^d + a_1(X)Y^{d-1} + \cdots + a_d(X) \in \mathbb{K}[X, Y]$ *be a polynomial such that* $D(X) = \mathrm{disc}_Y F(X, Y) \neq 0$. *For any* $Q = (a_1, \dots, a_n, b) \in V(F)$ *we put* $F_Q(X, Y) = F(a_1 + X, \dots, a_n + X_n, b + Y)$ (*therefore $F_Q(X, Y)$ is distinguished in Y with order* $\mathrm{mult}_Q \pi$). *Then for every $P \in \mathbb{K}^n$, we have*

$$\mathrm{ord}_P D = \sum_{Q \in \pi^{-1}(P)} \tilde{\mu}(F_Q) + d - \#\pi^{-1}(P).$$

Proof. Let $P = (a_1, \dots, a_n)$ and $r = \#\pi^{-1}(P)$. Then $\pi^{-1}(P) = \{Q_1, \dots, Q_r\}$, where $Q_i = (P, b_i)$ with $b_i \neq b_j$ for $i \neq j$. Let $d_i = \mathrm{mult}_{Q_i}\pi = \mathrm{ord}_{b_i} F(a, Y)$. Then $F(P, Y) = (Y - b_1)^{d_1} \dots (Y - b_r)^{d_r}$. We have to check that

$$\mathrm{ord}_P D = \sum_{i=1}^{r} \tilde{\mu}(F_{(P,b_i)}) + d - r. \tag{1}$$

Let $F_P(X, Y) = F(a_1 + X_1, \dots, a_n + X_n, Y)$ and $D_P(X) = D(a_1 + X_1, \dots, a_n + X_n)$. Then $D_P(X) = \mathrm{disc}_Y F_P(X, Y)$ and $\mathrm{ord}_P D(X) = \mathrm{ord}_0 D(X)$. Moreover, $(F_P)_{(0,b_i)} = F_{(P,b_i)}$ for $i = 1, \dots, r$, and it suffices to prove (1) for F_P at $0 \in \mathbb{K}^n$. Henceforth, we assume that $P = 0 \in \mathbb{K}^n$. First let us consider the case $r = 1$. Then (1) reduces (for $P = 0$) to the formula

$$\mathrm{ord}_0 D = \tilde{\mu}(F_{(0,b_1)}) + d - 1 \qquad \text{provided that } F(0, Y) = (Y - b_1)^d. \tag{2}$$

The polynomials $F_{(0,b_1)}(X, Y) = F(X, b_1 + Y)$ and $F(X, Y)$ have the same Y-discriminant, and hence (2) follows directly from the definition of $\tilde{\mu}$. Suppose that $r > 1$. By Hensel's Lemma, we have

$$F(X, Y) = \prod_{i=1}^{r} F_i(X, Y) \qquad \text{in } \mathbb{K}[[X]][Y] \qquad \text{with } F(0, Y) = (Y - b_i)^{d_i}.$$

Let $D_i(X) = \mathrm{disc}_Y F_i(X, Y)$ for $i = 1, \dots, r$ and $R_{ij}(X) = Y$-resultant $F_i(X, Y), F_j(X, Y)$ for $i \neq j$. By the product formula for the discriminant

$$D(X) = \prod_{i=1}^{r} D_i(X) \prod_{1 \le i < j \le r} R_{ij}(X)^2,$$

we get

$$\mathrm{ord}_0 D(X) = \sum_{i=1}^{r} \mathrm{ord}_0 D_i(X)$$

since $R_{ij}(0) = (b_i - b_j)^{d_i d_j} \neq 0$ for $i \neq j$.

By Formula (2) (which applies to the polynomials with coefficients in $\mathbb{K}[[X]]$), we have

$$\mathrm{ord}_0 D_i = \tilde{\mu}((F_i)_{(0,b_i)}) + d_i - 1 \qquad \text{for } i = 1, \dots, r.$$

Since $(F_i)_{(0,b_i)}$ and $F_{(0,b_i)}$ are associated in $\mathbb{K}[[X, Y]]$, we can write

$$\mathrm{ord}_0 D_i = \tilde{\mu}(F_{(0,b_i)}) + d_i - 1$$

and

$$\mathrm{ord}_0 D = \sum_{i=1}^{r} (\tilde{\mu}(F_{(0,b_i)}) + d_i - 1) = \sum_{i=1}^{r} \tilde{\mu}(F_{(0,b_i)}) + d - r. \qquad \square$$

Proof of Theorem 1.1. Theorem 1.1 follows directly from Lemmas 3.1 and 3.3. □

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