On computational properties of decomposable semigroups

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Sketch of the talk

- Preliminaries.
- Is a semigroup a decomposable semigroup?
- Some properties of the ideal associated to a decomposable semigroup.
- Decomposable variety.
- Algebraic Statistics.

Definition

$D \in \mathbb{Z}^{m\times n}$ is a HNF-diagonal matrix if:

1. the null rows are on the bottom side of the matrix,
2. the null columns are on the right side of the matrix,
3. every block D_i has maximal rank, it is in Hermite normal form and for each disjoint partition of $F(D_i)$, $B_1 \sqcup B_2$, we have:

$$\left(\bigcup_{f\in F(B_1)} \mathrm{supp}\,(f)\right)\bigcap\left(\bigcup_{f\in F(B_2)} \mathrm{supp}\,(f)\right) \neq \emptyset.$$

Example

$$\begin{pmatrix} 1 & 0 & -1 & 2 & 0 & 0 & 0 \\ 0 & 4 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

is a HNF-diagonal matrix, where

$$D_1 = \begin{pmatrix} 1 & 0 & -1 & 2 \\ 0 & 4 & 4 & 0 \end{pmatrix} \quad \text{and} \quad D_2 = \begin{pmatrix} 3 & 1 & 2 \end{pmatrix}.$$

Definition

$L \in \mathbb{Z}^{m\times n}$ is HNF-diagonalizable if there exists an unimodular matrix P and a column permutation matrix Q such that PLQ is a HNF-diagonal matrix.

Lemma

$L \in \mathbb{Z}^{m\times n}$ is HNF-diagonalizable iff there exists a column permutation matrix Q and a HNF-diagonal matrix D such that

$$rowspan_{\mathbb{Z}}(L) \equiv rowspan_{\mathbb{Z}}(LQ) = rowspan_{\mathbb{Z}}(D) =$$

$$rowspan_{\mathbb{Z}}(D_1) \times \cdots \times rowspan_{\mathbb{Z}}(D_t).$$

Lemma

There exists an algorithm to determine if L is HNF-diagonalizable. This algorithm finds two matrices Q and P such that PLQ is a HNF-diagonal matrix.

Example

Algorithm HNF-diagonalization

$$L = \begin{pmatrix} -1 & -4 & 12 & 4 & -3 & -2 & 8 \\ -4 & -4 & -9 & -3 & 0 & -8 & -6 \\ 7 & 8 & 12 & 4 & 1 & 14 & 8 \\ 7 & 16 & 6 & 2 & 9 & 14 & 4 \\ -1 & -4 & -3 & -1 & -3 & -2 & -2 \end{pmatrix}$$

$$\downarrow$$

$$\mathrm{HNF}(L) = \begin{pmatrix} 1 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 4 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 3 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

$$\downarrow$$

$$\mathrm{HNF}\left[\begin{pmatrix} -1 & -4 & 12 & 4 & -3 & -2 & 8 \\ -4 & -4 & -9 & -3 & 0 & -8 & -6 \\ 7 & 8 & 12 & 4 & 1 & 14 & 8 \\ 7 & 16 & 6 & 2 & 9 & 14 & 4 \\ -1 & -4 & -3 & -1 & -3 & -2 & -2 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}\right] =$$

$$= \begin{pmatrix} 1 & 0 & -1 & 2 & 0 & 0 & 0 \\ 0 & 4 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Lemma

Let S be a decomposable semigroup

$$I_S = I_{S_1} + I_{S_2} + \cdots + I_{S_t}$$

$$\Updownarrow$$

$$\ker S \equiv \ker S_1 \times \ker S_2 \times \cdots \times \ker S_t$$

Theorem

Let S be a semigroup and L a matrix such that $\ker S = rowspan_{\mathbb{Z}}(L)$. Then S is a decomposable semigroup iff L is a HNF-diagonalizable matrix.

Remark

If the semigroup S is torsion free we can study the decomposition applying the algorithm to the matrix whose columns are formed by a system of generators of S. In this case we obtain the decomposition operating directly in S.

Remark

The algorithm can be use to obtain a nice system generators *of a semigroup isomorphic to S.*

$I_S = I_{S_1} + I_{S_2} + \cdots + I_{S_t}$ be the decomposition of I_S.

Proposition

Then the Markov basis/Gröbner basis/Graver basis/universal Gröbner basis of I_S is the disjoint union of the Markov basis/Gröbner basis/Graver basis/universal Gröbner basis of each I_{S_i}.

Corollary

Are equivalent:

1. *I_S has a unique Markov basis (up to scalar multiple of its elements).*
2. *I_{S_i} has a unique Markov basis, for all $i = 1, \ldots, t$.*

Proposition

Let I_S is a complete intersection iff I_{S_i} is a complete intersection for all $i = 1, \ldots, t$.

Proposition

S is a gluing iff there exists $i \in \{1, \ldots, t\}$ such that S_i is a gluing of semigroups.

Notation

Let $S = \langle a_1, \dots, a_n \rangle \subset \mathbb{Z}^m$ be a torsion free semigroup.

$$V(I_S) = \{x \in \Bbbk^n | f(x) = 0, \forall f \in I_S\}$$

corresponds with the parametrized surfaced by the equations $x_i = l^{a_i}$, with $i = 1, \dots, n$ and $l \in (\Bbbk^*)^m$.

Definition

If $S = S_1 \oplus \cdots \oplus S_t$ is decomposable semigroup, then

$$V(I_S) = V(I_{S_1}) \cap V(I_{S_2}) \cap \cdots \cap V(I_{S_t}),$$

where all the varieties $V(I_{S_i}) \subset \Bbbk^{\mathrm{card}(A_i)}$ are embedded in $\Bbbk^n$.
We say that a variety is a decomposable variety if it is obtained from a decomposable semigroup.

Example

Consider the toric variety, V, determined by the parametrization

$$\left\{\begin{array}{lllllll} x_1 & = & l_1^{-1} & l_2^{-4} & l_3^{7} & l_4^{7} & l_5^{-1} \\ x_2 & = & l_1^{-4} & l_2^{-4} & l_3^{8} & l_4^{16} & l_5^{-4} \\ x_3 & = & l_1^{12} & l_2^{-9} & l_3^{12} & l_4^{6} & l_5^{-3} \\ x_4 & = & l_1^{4} & l_2^{-3} & l_3^{4} & l_4^{2} & l_5^{-1} \\ x_5 & = & l_1^{-3} & & l_3 & l_4^{9} & l_5^{-3} \\ x_6 & = & l_1^{-2} & l_2^{-8} & l_3^{14} & l_4^{14} & l_5^{-2} \\ x_7 & = & l_1^{8} & l_2^{-6} & l_3^{8} & l_4^{4} & l_5^{-2} \end{array}\right. ,$$

and let $S \subset \mathbb{Z}^5$ be the semigroup associated

$$\begin{pmatrix} -1 & -4 & 12 & 4 & -3 & -2 & 8 \\ -4 & -4 & -9 & -3 & 0 & -8 & -6 \\ 7 & 8 & 12 & 4 & 1 & 14 & 8 \\ 7 & 16 & 6 & 2 & 9 & 14 & 4 \\ -1 & -4 & -3 & -1 & -3 & -2 & -2 \end{pmatrix}.$$

Example

$$S = \left\langle \begin{pmatrix} -1 & -4 & -3 & -2 \\ -4 & -4 & 0 & -8 \\ 7 & 8 & 1 & 14 \\ 7 & 16 & 9 & 14 \\ -1 & -4 & -3 & -2 \end{pmatrix} \right\rangle \bigoplus \left\langle \begin{pmatrix} 12 & 4 & 8 \\ -9 & -3 & -6 \\ 12 & 4 & 8 \\ 6 & 2 & 4 \\ -3 & -1 & -2 \end{pmatrix} \right\rangle,$$

and S is isomorphic to the direct sum of the semigroups S_1 and S_2,

$$\left\langle \begin{pmatrix} 1 & 0 & -1 & 2 \\ 0 & 4 & 4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right\rangle \bigoplus \left\langle \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 3 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\rangle$$

Example

Using this decomposition we can give *nice reparametrization* of the affine toric variety V,

$$\left\{\begin{array}{lllllll} x_1 & = & l_1^{-1} & l_2^{-4} & l_3^{7} & l_4^{7} & l_5^{-1} \\ x_2 & = & l_1^{-4} & l_2^{-4} & l_3^{8} & l_4^{16} & l_5^{-4} \\ x_3 & = & l_1^{12} & l_2^{-9} & l_3^{12} & l_4^{6} & l_5^{-3} \\ x_4 & = & l_1^{4} & l_2^{-3} & l_3^{4} & l_4^{2} & l_5^{-1} \\ x_5 & = & l_1^{-3} & & l_3 & l_4^{9} & l_5^{-3} \\ x_6 & = & l_1^{-2} & l_2^{-8} & l_3^{14} & l_4^{14} & l_5^{-2} \\ x_7 & = & l_1^{8} & l_2^{-6} & l_3^{8} & l_4^{4} & l_5^{-2} \end{array}\right. \to \left\{\begin{array}{lllll} x_1 & = & q_1 & & \\ x_2 & = & & q_2^4 & \\ x_3 & = & & & q_3^3 \\ x_4 & = & & & q_3 \\ x_5 & = & q_1^{-1} & q_2^4 & \\ x_6 & = & q_1^2 & & \\ x_7 & = & & & q_3^2 \end{array}\right. .$$

Futhermore,

$$I_S = \left\langle x_1^2 - x_6,\, x_1x_5 - x_2 \right\rangle + \left\langle x_4^2 - x_7,\, x_4^3 - x_3 \right\rangle,$$

and

$$V = V\left(\left\langle x_1^2 - x_6,\, x_1x_5 - x_2 \right\rangle\right) \cap V\left(\left\langle x_4^2 - x_7,\, x_4^3 - x_3 \right\rangle\right).$$

Example

$$A = \begin{pmatrix}
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
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0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&1
\end{pmatrix}$$

Example

A is HNF-diagonalizable and its associated HNF-diagonal matrix is

$$D = \begin{pmatrix} D_1 & \ominus & \ominus & \ominus & \ominus & \ominus & \ominus & \ominus \\ \ominus & D_1 & \ominus & \ominus & \ominus & \ominus & \ominus & \ominus \\ \ominus & \ominus & D_1 & \ominus & \ominus & \ominus & \ominus & \ominus \\ \ominus & \ominus & \ominus & D_1 & \ominus & \ominus & \ominus & \ominus \\ \ominus & \ominus & \ominus & \ominus & D_1 & \ominus & \ominus & \ominus \\ \ominus & \ominus & \ominus & \ominus & \ominus & D_1 & \ominus & \ominus \\ \ominus & \ominus & \ominus & \ominus & \ominus & \ominus & D_1 & \ominus \\ \ominus & \ominus & \ominus & \ominus & \ominus & \ominus & \ominus & D_1 \\ \ominus & \ominus & \ominus & \ominus & \ominus & \ominus & \ominus & \ominus \end{pmatrix},$$

where $D_1 = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$, and the permutation matrix Q swaps the columns of A grouping them in the following sets:

$$A_1 = \{1, 9, 17, 25\},\ A_2 = \{2, 10, 18, 26\},\ A_3 = \{3, 11, 19, 27\},$$

$$A_4 = \{4, 12, 20, 28\},\ A_5 = \{5, 13, 21, 29\},\ A_6 = \{6, 14, 22, 30\},$$
$$A_7 = \{7, 15, 23, 31\},\ A_8 = \{8, 16, 24, 32\}.$$

Example

We obtain:

- the Markov basis of $I_{\langle A_1\rangle} = I_{\langle D_1\rangle} \subset \Bbbk[x_1, x_9, x_{17}, x_{25}]$ is $\{x_1x_{25} - x_9x_{17}\}$,
- this basis is unique,
- $I_{\langle A_1\rangle}$ is a complete intersection,
- $\langle D_1\rangle$ is the gluing of the semigroups $S_1' = \langle(1,0,0),(-1,1,1)\rangle$ and $S_2' = \langle(0,1,0),(0,0,1)\rangle$, and then $\langle A_1\rangle$ is the gluing of the semigroups generated by the columms $\{1, 25\}$ and $\{9, 17\}$ of A.

Then the ideal $I_{\langle A\rangle} \subset \Bbbk[x_1, \ldots, x_{32}]$ verifies:

- has a unique Markov basis,

$$\{x_1x_{25} - x_9x_{17},\ x_2x_{26} - x_{10}x_{18},\ x_3x_{27} - x_{11}x_{19},\ x_4x_{28} - x_{12}x_{20},\ x_5x_{29} - x_{13}x_{21}$$

$$x_6x_{30} - x_{14}x_{22},\ x_7x_{31} - x_{15}x_{23},\ x_8x_{32} - x_{16}x_{24}\},$$

- it is a complete intersection.

http://markov-bases.de/

	N^O of semigroups	N^O of generators	Unique
BM4r2-2_bin	2	8–8	no
G15g_bin	2	8–8	YES
G17g_bin	4	4–4–4–4	YES
5-4m1_bin	2	16–16	yes
BM5r3-2_bin	2	16–16	no
BM5r3-4_bin	2	16–16	no
BPg_bin	8	4-4-4-4-4-4-4-4	YES
G34g_bin	2	16–16	NO
G40g_bin	2	16–16	NO
G42g_bin	2	16–16	YES
G45g_bin	2	16–16	YES
G46g_bin	4	8-8-8-8	no
G47g_bin	2	16-16	YES
SPg_bin	2	16–16	YES
6-1I2_bin	4	16-16-16-16	yes
G111g_bin	2	32–32	NO
G117g_bin	2	32–32	NO
G133g_bin	2	32–32	NO
G135g_bin	2	32–32	NO
G136g_bin	2	32–32	NO
G144g_bin	2	32–32	NO
G156g_bin	2	32–32	NO
G158g_bin	2	32–32	NO
G161g_bin	4	16-16-16-16	NO
G162g_bin	2	32–32	NO
G164g_bin	2	32–32	NO
G165g_bin	2	32–32	YES
G92g_bin	2	32–32	NO

"no/yes" uniqueness solved in *DataBase*.
"NO/YES" uniqueness solved in this work.

	Graver B. without HNF-D.	Time HNF-D.	Graver B. with HNF-D.
G111g_bin	6 days + 03:36:15	0:00:07	2 days + 09:28:44
G117g_bin	23:02:04	0:00:09	6:08:52
G135g_bin	5 days + 09:59:07	0:00:09	1 day + 11:01:10
G136g_bin	4 days + 10:25:37	0:00:09	1 day + 02:46:15
G144g_bin	1 day + 18:40:46	0:00:09	12:25:12
G156g_bin	0:01:31	0:00:10	0:00:26
G158g_bin	5 days + 16:24:45	0:00:10	1 day + 15:11:57
G162g_bin	4 days + 04:53:47	0:00:14	1 day + 12:28:55
G164g_bin	2 days + 20:54:13	0:00:12	1 day + 00:36:59
G92g_bin	2 days + 02:22:39	0:00:09	16:58:12

Table : Time table of Graver bases (dd + hh:mm:ss).

http://www.uca.es/dpto/C101/pags-personales/alberto.vigneron/hnfdecomposable.zip

File name	Matrix dim.	Size of entries	Time without HNF-D.	Time of HNF-D.	Time with HNF-D.
11pru_1.mat	39x40	(-51,55)	0:00:00	0.23 s	0.00 s
11pru_7.mat	38x47	(-54,55)	1:07:43	0.18 s	0.00 s
11pru_5.mat	41x47	(-46,45)	aborted after 7 days	0.18 s	0.00 s
11pru_8.mat	35x49	(-50,45)	aborted after 7 days	0.20 s	0.00 s
07pru_0.mat	39x38	(-250,287)	0:00:31	0.20 s	0.00 s
07pru_4.mat	39x43	(-259,275)	5:29:51	0.21 s	0.00 s
06pru_4.mat	34x39	(-366,370)	aborted after 7 days	0.17 s	0.00 s
06pru_7.mat	36x37	(-424,413)	0:01:01	0.17 s	0.00 s
01pru_37.mat	56x53	(-551,682)	5:38:52	0.48 s	0.00 s
01pru_9.mat	61x63	(-556,621)	out of memory 1 day + 16:49:13	0.65 s	0.00 s
00pru_0.mat	83x66	(-998,1081)	out of memory 08:59:15	1.3 s	0.00 s
00pru_1.mat	69x76	(-1088,851)	out of memory 1 day + 13:29:33	1.0 s	0.00 s

Table : Time table of Gröbner bases (dd + hh:mm:ss).

Grupo de investigación ASCA
Álgebras de Semigrupos, Computación y Aplicaciones

Thanks for your attention!