Jumping numbers and multiplier ideals for complete ideals of 2-dimensional regular local rings.

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(based on joint works with Francisco Monserrat and Fernando Hernando)

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Talk's objectives

- $\mathfrak{a}$: simple complete ideal of local ring at a closed point of a smooth complex algebraic surface.
- $P_{\mathfrak{a}}(t)$: Poincaré series associated with the sequence of multiplier ideals of $\mathfrak{a}$.
- Objective 1: to show that $P_{\mathfrak{a}}(t)$ is "rational".
- Objective 2: to obtain a formula for the log-canonical threshold of a reduced germ of a plane curve.

Based on:

Objective 1: C. G. & F. Monserrat, *The Poincaré series of multiplier ideals of a simple complete ideal in a local ring of a smooth surface*, *Advances in Mathematics* **225** (2010), 1046-1068.

Objective 2: C. G., F. Hernando & F. Monserrat, *The log-canonical threshold of a plane curve*, ArXiv:1211.6274v1.

Multiplier ideal sheaf of an ideal sheaf

- $\mathfrak{a}$: ideal sheaf on a smooth variety Y over $\mathbb{C}$.
- *log resolution* $\pi : X \longrightarrow Y$ of $\mathfrak{a}$: birational, proper, X smooth,
$$\mathfrak{a} \cdot \mathcal{O}_X \;=\; \mathcal{O}_X(-D),$$
where D is an effective divisor on X such that $D + \mathrm{except}(\pi)$ has SNC support.

Definition

Let $\lambda \in \mathbb{Q}_{>0}$. *Multiplier ideal* sheaf of $\mathfrak{a}$ with coefficient λ:

$$\mathcal{J}(\mathfrak{a}^{\lambda}) := \pi_* \mathcal{O}_X(K_{X/Y} - \lfloor \lambda D \rfloor).$$

Notation: $\lfloor \sum a_i E_i \rfloor := \sum \lfloor a_i \rfloor E_i, \quad K_{X/Y} \equiv K_X - \pi^* K_Y$.

Independent of the log resolution

"Good" properties of multiplier ideals

Multiplier ideals have become an important tool in algebraic geometry due to their interesting properties:

- Vanishing theorems.
- Multiplier ideals provide a "measure" of the singularities.
- Analytic description: If Y is a smooth affine variety and $\mathfrak{a} = \langle f_1, f_2, \ldots, f_r \rangle \subseteq \mathcal{O}_Y(Y)$:

$$\mathcal{J}(\mathfrak{a}^{\lambda})^{an} =_{\text{locally}} \left\{ h \text{ holomorphic} \mid \frac{|h|^2}{(\sum |f_i|^2)^{\lambda}} \text{ is locally integrable} \right\}.$$

"Smaller" multiplier ideals correspond to "worse" singularities.

Reference: Positivity in Algebraic Geometry II (R. Lazarsfeld).

- Multiplier ideals are integrally closed (or complete).
- In dimension 2, every integrally closed ideal is a multiplier ideal (Lipman-Watanabe (2003), Favre-Jonsson (2005)).
- False in general (Lazarsfeld-Lee (2007)).
- Examples of multiplier ideals are hard to compute. Very few cases are known:
  - Multiplier ideals of monomial ideals (Howald, 2001).
  - Multiplier ideals of hyperplane arrangements (Mustaţă, 2006) (related works: N. Budur and M. Saito).

Jumping numbers (i)

$Y := \mathrm{Spec}(R)$, where $R = \mathcal{O}_{Z,O}$, Z smooth complex surface.

$\mathfrak{a}$: ideal sheaf on Y; $\pi : X \to Y$ log resolution of $\mathfrak{a}$.

$\mathfrak{a} \cdot \mathcal{O}_X = \mathcal{O}_X(-D)$, $D = \sum_{i=1}^n b_i E_i$, $K_{X/Y} = \sum_{i=1}^n a_i E_i$ and

$$\mathcal{J}(\mathfrak{a}^\lambda) = \{h \in R \mid \mathrm{div}(\pi^* h) + K_{X/Y} - \lfloor \lambda D \rfloor \geq 0\} =$$
$$= \{h \in R \mid \nu_{E_i}(h) \geq \lfloor \lambda b_i \rfloor - a_i \ \ \forall i\}.$$

$\mathcal{J}(\mathfrak{a}^\lambda) = \mathcal{J}(\mathfrak{a}^{\lambda+\epsilon})$ for sufficiently small $\epsilon > 0$.

Jumping numbers (ii)

There exists an increasing sequence of rational numbers $0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots$ s.t. $\mathcal{J}(\mathfrak{a}^\lambda)$ are constant for $\lambda \in [\lambda_i, \lambda_{i+1}[$ and

$$R \supsetneq \mathcal{J}(\mathfrak{a}^{\lambda_1}) \supsetneq \mathcal{J}(\mathfrak{a}^{\lambda_2}) \supsetneq \cdots .$$

$\{\lambda_1, \lambda_2, \ldots\}$ are called *jumping numbers* (λ_1: log-canonical threshold).

Definition

$\lambda \in \mathbb{Q}_{>0}$ is a *candidate jumping number* from a prime exceptional divisor E_i if $\lambda b_i \in \mathbb{Z}$.

- The set of jumping numbers of $\mathfrak{a}$ is, in general, strictly contained in the set of non trivial candidate jumping numbers:

$$\left\{ \frac{a_i + m}{b_i} \mid m \in \mathbb{Z}_{>0} \right\}.$$

- Periodicity: If $\lambda > 2$,
  λ is a jumping number $\Leftrightarrow$ $\lambda - 1$ is a jumping number.

Objective 1

To compute the Poincaré series:

$$P_{\mathfrak{a}}(t) := \sum_{\lambda \in \mathcal{H}} \dim_{\mathbb{C}} \left(\frac{\mathcal{J}(\mathfrak{a}^{\lambda^-})}{\mathcal{J}(\mathfrak{a}^{\lambda})} \right) t^{\lambda},$$

when $\mathfrak{a}$ is a simple complete ideal of R and $\mathcal{H}$ is its set of jumping numbers.

- λ^- being the largest jumping number less than λ.

Expressions of the jumping numbers (i)

There exists a one to one correspondence among

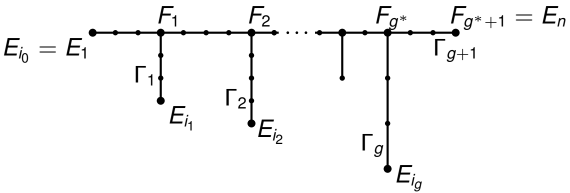
- Simple complete ideals $\mathfrak{a}$.
- Plane divisorial valuations ν of the fraction field of R (centered at R).
- Finite simple sequences of point blowing-ups:

$$X = X_n \to X_{n-1} \to \cdots \to X_1 \to \mathrm{Spec}(R)$$

(log resolution of $\mathfrak{a}$).

Expressions of the jumping numbers (ii)

ν: divisorial valuation defining a simple complete ideal $\mathfrak{a}$ of R.
If $\mathfrak{a} \cdot \mathcal{O}_X = \mathcal{O}_X(-D)$: $D = \sum_{j=1}^{n} b_j E_j = \sum_{j=1}^{n} \nu(\varphi_j) E_j$.
φ_j: General element for $E_j \equiv$ element of R giving an equation of an analytically irreducible germ of curve whose strict transform on X_j is smooth and intersects E_j transversally at a non-singular point of the exceptional locus.



g: Number of Puiseux pairs; g^*: Number of "star vertices".

Maximal contact values:

$\bar{\beta}_k = \nu(\varphi_{i_k}) \quad k = 0, 1, \ldots, g, \ \bar{\beta}_{g+1} = \nu(\varphi_n) = \nu(\mathfrak{a})$.

Expressions of the jumping numbers (iii)

Jumping numbers

If $\mathfrak{a}$ is a simple complete ideal, the set of jumping numbers of $\mathfrak{a}$ is $\mathcal{H} = \cup_{i=1}^{g^*+1} \mathcal{H}_i$, where

$$\mathcal{H}_i := \left\{ \lambda(i,p,q,r) := \frac{p}{e_{i-1}} + \frac{q}{\bar{\beta}_i} + \frac{r}{e_i} \;\middle|\; \frac{p}{e_{i-1}} + \frac{q}{\bar{\beta}_i} \leq \frac{1}{e_i}; p,q \geq 1, r \geq 0 \right\}$$

whenever $1 \leq i \leq g^*$, and

Expressions of the jumping numbers (iv)

Jumping numbers

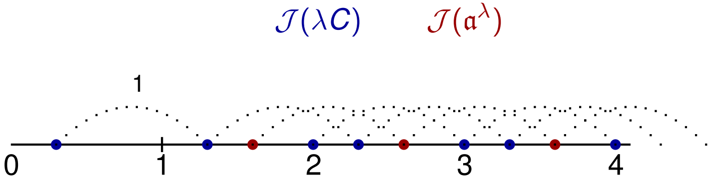
$$\mathcal{H}_{g^*+1} := \left\{ \lambda(g^*+1, p, q) := \frac{p}{e_{g^*}} + \frac{q}{\bar{\beta}_{g^*+1}} \;\middle|\; p, q \geq 1 \right\},$$

p, q and r being integer numbers and $e_i = \gcd(\bar{\beta}_0, \ldots, \bar{\beta}_i)$.

T. Järvilehto, Jumping numbers of a simple complete ideal in a two-dimensional regular local ring. *Mem. Amer. Math. Soc.* **214** (2011). Also (curve case): D. Naie (2009), K. Tucker (PhD. dissertation, 2010).

Jumping numbers of a general curve of $\mathfrak{a}$

C : General curve of $\mathfrak{a}$, defined by φ_n.



$$\mathcal{H}_C := \{\text{Jumping numbers of } C\}.$$

$$\Omega := \{\lambda \in]1,2[\ \mid \ \lambda \text{ is a j.n. of } \mathfrak{a} \text{ and } \lambda - 1 \text{ is not a j.n.}\} \subseteq \mathcal{H}_{g^*+1}.$$

$$\mathcal{H} = (\mathcal{H}_C \setminus \{1\}) \cup \{\lambda + k \mid \lambda \in \Omega \ \wedge \ k \in \mathbb{Z}_{\geq 0}\}.$$

The multiplier ideal preceding a given one

$$\Delta := \{E_j \mid \lambda \text{ is a candidate jumping number from } E_j\}$$

$$\underbrace{\pi_*\mathcal{O}_X(K_{X/Y} - \lfloor \lambda D \rfloor) = \mathcal{J}(\mathfrak{a}^\lambda)}_{\{h\in R \mid \nu_{E_i}(h)\geq \lambda b_i - a_i \ \ \forall i\}} \subsetneq \underbrace{\mathcal{J}(\mathfrak{a}^{\lambda^-}) = \pi_*\mathcal{O}_X(K_{X/Y} - \lfloor \lambda D \rfloor + \sum_{E_j\in\Delta} E_j)}_{\{h\in R \mid \nu_{E_i}(h)\geq \lambda b_i - a_i \ \ \forall i\notin\Delta \ \wedge\ \nu_{E_i}(h)\geq \lambda b_i - a_i - 1 \ \ \forall i\in\Delta\}}$$

For each $E_j \in \Delta$:

$$\mathcal{J}(\mathfrak{a}^\lambda) \subseteq \underbrace{\pi_*\mathcal{O}_X(K_{X/Y} - \lfloor \lambda D \rfloor + E_j)}_{\{h\in R \mid \nu_{E_i}(h)\geq \lambda b_i - a_i \ \ \forall i\neq j \ \wedge \ \nu_{E_j}(h)\geq \lambda b_j - a_j - 1\}} \subseteq \mathcal{J}(\mathfrak{a}^{\lambda^-}).$$

Question: For which $E_j \in \Delta$

$$\mathcal{J}(\mathfrak{a}^\lambda) \subsetneq \pi_*\mathcal{O}_X(K_{X/Y} - \lfloor \lambda D \rfloor + E_j)\ ?$$

Contribution of divisors to candidate jumping numbers (i)

K.E. Smith, H.M. Thompson, Irrelevant exceptional divisors for curves on a smooth surface, Contemp. Math., vol. 448 (2007).

Definition

Let λ be a jumping number of $\mathfrak{a}$. E_j contributes λ whenever

1. λ is a candidate jumping number from E_j and
2. $\mathcal{J}(\mathfrak{a}^{\lambda}) \subsetneq \pi_* \mathcal{O}_X(K_{X/Y} - \lfloor \lambda D \rfloor + E_j)$.

Contribution of divisors to candidate jumping numbers (ii)

Criterion of contribution:

- Assume that λ is a jumping number. Then E_j contributes λ if and only if λ is a candidate jumping number from E_j and

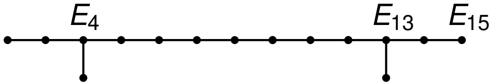
$$-\lfloor \lambda D \rfloor \cdot E_j \geq 2.$$

- Moreover, if E_j contributes some jumping number, then $E_j = F_i$ for some $i \in \{1, \ldots, g^* + 1\}$.

(K. Tucker (2008): Extension to surfaces with rational singularities).

Example

Let $\mathfrak{a} \subseteq R = \mathbb{C}[x,y]_{(x,y)}$ be the simple complete ideal associated with a divisorial valuation ν centered at R with maximal contact values $\bar{\beta}_0 = 6$, $\bar{\beta}_1 = 10$, $\bar{\beta}_2 = 45$ and $\bar{\beta}_3 = 92$.



$$D = 6E_1 + 10E_2 + 18E_3 + 30E_4 + 32E_5 + 34E_6 + 36E_7 + 38E_8 + 40E_9 + 42E_{10} + 44E_{11} + 45E_{12} + 90E_{13} + 91E_{14} + 92E_{15} = (6,10,18,30,32,34,36,38,40,42,44,45,90,91,92).$$

Jumping number: $\lambda = \frac{23}{30}$.

$\lambda \in \mathcal{H}_1 \cap \mathcal{H}_2$.

λ is candidate j. n. for: $E_4 = F_1$ and $E_{13} = F_2$.

Example (ii)

Recall: E divisor with exceptional support, E is antinef whenever $E \cdot E_j \leq 0$ for all j. We set $E^{\sim}$ the antinef closure, i.e., the least antinef divisor $\geq E$.
$\pi_* \mathcal{O}_X(-E) = \pi_* \mathcal{O}_X(-E^{\sim})$.

$\mathcal{J}(\mathfrak{a}^{\lambda}) = \pi_* \mathcal{O}_X(K_{X/Y} - \lfloor \lambda D \rfloor)$.

$\lfloor \lambda D \rfloor - K_{X/Y} = (3, 5, 9, 16, 16, 17, 17, 18, 18, 19, 19, 19, 39, 38, 38)$.

$(\lfloor \lambda D \rfloor - K_{X/Y})^{\sim} = (4, 6, 11, 18, 19, 20, 20, 20, 20, 20, 20, 20, 40, 40, 40)$.

Previous jumping number: $\lambda^- = \frac{67}{90}$.

$\mathcal{J}(\mathfrak{a}^{\lambda^-}) = \pi_* \mathcal{O}_X(K_{X/Y} - \lfloor \lambda^- D \rfloor) = \pi_* \mathcal{O}_X(K_{X/Y} - \lfloor \lambda D \rfloor + E_4 + E_{13}))$.

$\lfloor \lambda^- D \rfloor - K_{X/Y} = (3, 5, 9, 15, 15, 16, 16, 17, 17, 18, 18, 18, 37, 36, 36)$.

$(\lfloor \lambda^- D \rfloor - K_{X/Y})^{\sim} = (3, 5, 9, 15, 16, 17, 18, 19, 19, 19, 19, 19, 38, 38, 38)$.

Example (iii)

$(\lfloor \lambda D \rfloor - K_{X/Y})^{\sim} = (4, 6, 11, 18, 19, 20, 20, 20, 20, 20, 20, 20, 40, 40, 40).$

$(\lfloor \lambda^{-} D \rfloor - K_{X/Y})^{\sim} = (3, 5, 9, 15, 16, 17, 18, 19, 19, 19, 19, 19, 38, 38, 38).$

Does E_4 contribute λ?

$(\lfloor \lambda D \rfloor - K_{X/Y} - E_4)^{\sim} = (3, 5, 9, 15, 16, 17, 18, 19, 20, 20, 20, 20, 40, 40, 40).$

Then $\mathcal{J}(\mathfrak{a}^{\lambda}) \neq \pi_* \mathcal{O}_X(K_{X/Y} - \lfloor \lambda D \rfloor + E_4) \Rightarrow$ E_4 contributes λ.

But: $\mathcal{J}(\mathfrak{a}^{\lambda^{-}}) \neq \pi_* \mathcal{O}_X(K_{X/Y} - \lfloor \lambda D \rfloor + E_4).$

Does E_{13} contribute λ?
$(\lfloor \lambda D \rfloor - K_{X/Y} - E_{13})^{\sim} = (4, 6, 11, 18, 19, 19, 19, 19, 19, 19, 19, 19, 38, 38, 38).$

Then $\mathcal{J}(\mathfrak{a}^{\lambda}) \neq \pi_* \mathcal{O}_X(K_{X/Y} - \lfloor \lambda D \rfloor + E_{13}) \Rightarrow$ E_{13} contributes λ.

But: $\mathcal{J}(\mathfrak{a}^{\lambda^{-}}) \neq \pi_* \mathcal{O}_X(K_{X/Y} - \lfloor \lambda D \rfloor + E_{13}).$

Determination of the divisors contributing a jumping number

Theorem 1

A jumping number λ of a simple complete ideal $\mathfrak{a}$ belongs to the set $\mathcal{H}_i$ $(1 \leq i \leq g^* + 1)$ if and only if the prime exceptional divisor F_i contributes λ.

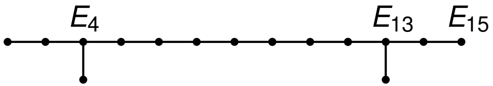
Corollary

The prime exceptional divisors that contribute a jumping number λ of a simple complete ideal $\mathfrak{a}$ are those divisors F_i such that $\lambda \in \mathcal{H}_i$.

Other independent proofs (curve case):
(D. Naie, 2009), (K. Tucker, PhD dissertation, 2010).

Example

Consider $\mathfrak{a}$ as in the first example.



Jumping number: $\lambda = \frac{7}{10} \in \mathcal{H}_2$.

λ is candidate j. n. for: $\{E_2, E_4 = F_1, E_9$ and $E_{13} = F_2\}$.

The unique divisor contributing λ is $E_{13} = F_2$.

$\lambda^- = \frac{61}{90}$.

$\mathcal{J}(\mathfrak{a}^{\lambda^-}) = \pi_* \mathcal{O}_X(K_{X/Y} - \lfloor \lambda D \rfloor + E_2 + E_4 + E_9 + E_{13})$.

But it can be checked that

$$\mathcal{J}(\mathfrak{a}^{\lambda^-}) = \pi_* \mathcal{O}_X(K_{X/Y} - \lfloor \lambda D \rfloor + F_2).$$

The multiplier ideal preceding a given one

Let λ be a jumping number of $\mathfrak{a}$.

$\Delta := \{E_j \mid \lambda \text{ is a candidate jumping number for } E_j\}$.

$\Delta' := \{E_j \mid E_j \text{ contributes } \lambda\} = \{F_i \mid \lambda \in \mathcal{H}_i\} \subseteq \Delta$.

$\mathcal{J}(\mathfrak{a}^{\lambda^-}) = \pi_* \mathcal{O}_X(K_{X/Y} - \lfloor \lambda D \rfloor + \sum_{E_j \in \Delta} E_j)$.

Question:

Is it true that $\mathcal{J}(\mathfrak{a}^{\lambda^-}) = \pi_* \mathcal{O}_X(K_{X/Y} - \lfloor \lambda D \rfloor + \sum_{F_i \in \Delta'} F_i)$?

In the affirmative case, one can "control" $\mathcal{J}(\mathfrak{a}^{\lambda^-})$ from

- $\mathcal{J}(\mathfrak{a}^{\lambda})$ and
- the indices i such that $\lambda \in \mathcal{H}_i$.

Answer

Theorem 2

Let λ be a jumping number of a simple complete ideal $\mathfrak{a}$. Then

$$\pi_*\mathcal{O}_X\left(K_{X/Y} - \lfloor \lambda D \rfloor + \sum_{l=1}^{s} F_{i_l}\right) = \mathcal{J}\left(\mathfrak{a}^{\lambda^-}\right),$$

where $\{i_1, i_2, \ldots, i_s\}$ is the set of indexes i, $1 \leq i \leq g^* + 1$, such that $\lambda \in \mathcal{H}_i$.

The Poincaré series

Theorem 3

The Poincaré series $P_{\mathfrak{a}}(t) = \sum_{\lambda \in \mathcal{H}} \dim \left(\frac{\mathcal{J}(\mathfrak{a}^{\lambda^-})}{\mathcal{J}(\mathfrak{a}^{\lambda})} \right) t^{\lambda}$ has this expression:

$$P_{\mathfrak{a}}(t) = \frac{1}{1-t} \sum_{i=1}^{g^*} \sum_{\lambda \in \mathcal{H}_i, \lambda < 1} t^{\lambda} + \frac{1}{(1-t)^2} \sum_{\lambda \in \Omega} t^{\lambda},$$

where

$$\Omega := \{\lambda \in \mathcal{H}_{g^*+1} \mid \lambda \leq 2 \text{ and } \lambda - 1 \notin \mathcal{H}_{g^*+1}\}.$$

Key facts of the proof

Lemma

$$\dim \frac{\mathcal{J}(\mathfrak{a}^{\lambda^-})}{\mathcal{J}(\mathfrak{a}^{\lambda})} = \dim \frac{\pi_* \mathcal{O}_X(K_{X|X_0} - \lfloor \lambda D \rfloor + \sum_{\lambda \in \mathcal{H}_i} F_i)}{\mathcal{J}(\mathfrak{a}^{\lambda})} =$$

$$= \sum_{\lambda \in \mathcal{H}_i} \underbrace{\dim \frac{\pi_* \mathcal{O}_X(K_{X|X_0} - \lfloor \lambda D \rfloor + F_i)}{\mathcal{J}(\mathfrak{a}^{\lambda})}}_{d_\lambda^i} = \sum_{\lambda \in \mathcal{H}_i} d_\lambda^i .$$

Consequence:

$$P_{\mathfrak{a}}(t) = \sum_{\lambda \in \mathcal{H}} \dim \left(\frac{\mathcal{J}(\mathfrak{a}^{\lambda^-})}{\mathcal{J}(\mathfrak{a}^{\lambda})} \right) t^{\lambda} = \sum_{i=1}^{g^*+1} P_i(t),$$

where $P_i(t) := \sum_{\lambda \in \mathcal{H}_i} d_\lambda^i t^\lambda$.

Computation of $P_i(t) = \sum_{\lambda \in \mathcal{H}_i} d^i_\lambda t^\lambda$

- $\lambda \in \mathcal{H}_i$ is a "primitive" element of $\mathcal{H}_i$ if $\lambda - 1 \notin \mathcal{H}_i$.
- When $1 \leq i \leq g^*$: $\{$Primitive elements of $\mathcal{H}_i\} = \mathcal{H}_i \cap]0,1[$
  $\mathcal{H}_i = \{\lambda + n \mid \lambda \in \mathcal{H}_i \cap]0,1[\text{ and } n \in \mathbb{N}\}$.

$$\left.\begin{array}{l} d^i_\lambda = 1 \text{ whenever } \lambda \in \mathcal{H}_i \cap]0,1[\\ \text{If } \lambda \in \mathcal{H}_i \cap]0,1[\Rightarrow d^i_{\lambda+n} = d^i_\lambda \ \ \forall n \end{array}\right\} \Rightarrow d^i_\lambda = 1 \ \ \forall \lambda \in \mathcal{H}_i.$$

- When $i = g^* + 1$:
  $\{$Primitive elements of $\mathcal{H}_{g^*+1}\} = \Omega$
  $= \{\lambda \in \mathcal{H}_{g^*+1} \mid \lambda \leq 2 \text{ and } \lambda - 1 \notin \mathcal{H}_{g^*+1}\}$
  $\mathcal{H}_{g^*+1} = \{\lambda + n \mid \lambda \in \Omega \text{ and } n \in \mathbb{N}\}$.

$$\left.\begin{array}{l} d^i_\lambda = 1 \text{ whenever } \lambda \in \Omega \\ \text{If } \lambda \in \Omega \Rightarrow d^i_{\lambda+n} = d^i_\lambda + n \ \ \forall n. \end{array}\right\}$$

Computation of $P_i(t) = \sum_{\lambda \in \mathcal{H}_i} d^i_\lambda t^\lambda$

- When $1 \leq i \leq g^*$:

$$P_i(t) = P_i(z_i^{e_{i-1}\bar{\beta}_i}) = \frac{1}{1 - z_i^{e_{i-1}\bar{\beta}_i}} \sum_{(p,q,s)\in B} z_i^{p\bar{\beta}_i + qe_{i-1} + (s+e_i)\frac{e_{i-1}}{e_i}\bar{\beta}_i}$$

$$= \frac{1}{1-t} \sum_{\lambda \in \mathcal{H}_i, \lambda < 1} t^\lambda.$$

- When $i = g^* + 1$:

$$P_{g^*+1}(t) = P_{g^*+1}(z_{g^*+1}^{e_{g^*}\bar{\beta}_{g^*+1}}) = \frac{1}{(1 - z_{g^*+1}^{e_{g^*}\bar{\beta}_{g^*+1}})^2} \sum_{(s,q)\in T} z_{g^*+1}^{s\bar{\beta}_{g^*+1} + qe_{g^*}}$$

$$= \frac{1}{(1-t)^2} \sum_{\lambda \in \Omega} t^\lambda.$$

Objective 2: the log-canonical threshold

Log-canonical threshold: minimum jumping number.

Antecedents:

- Analitically irreducible germ of curve (Igusa, 1977; Järvilehto, 2007): $\frac{1}{\bar{\beta}_0} + \frac{1}{\bar{\beta}_1}$.
- Product of two analitically irreducible germs of curve (Kuwata, 1999).
- For a plane curve (any number of branches):
  - There exist suitable local coordinates such that lct is $1/t$, where (t, t) is the unique diagonal point of the Newton polygon (Artal, Cassou-Nogués, Luengo, Melle-Hernández, 2008).
  - There exist suitable local coordinates such that lct is the lct of the *term ideal* (Aprodu, Naie, 2010).
- Irreducible quasi-ordinary hypersurface singularity (Budur, González-Pérez, González-Villa).

Our goal: expression of the lct of a reduced plane curve (any number of branches) in terms of maximal contact values.

$R = k[[x,y]]$: Formal power series ring with coefficients over an algebraically closed field k.

C: REDUCED curve defined by $f = f_1 \cdots f_r$, $f_i \in R$ irreducible.

C_i: curve defined by f_i.

Log resolution of $\mathfrak{a} := (f) \subseteq R$ (composition of blow-ups):

$$\pi : X = X_m \xrightarrow{\pi_m} X_{m-1} \longrightarrow \cdots \longrightarrow X_1 \xrightarrow{\pi_1} Y = \operatorname{Spec}(R).$$

Set of centers (constellation): $\mathcal{C} := \{P_i\}_{i=1}^m$.

$\mathfrak{a} \cdot \mathcal{O}_X = \mathcal{O}_X(-D)$ with $D = \tilde{C}_1 + \cdots + \tilde{C}_r + b_1 E_1 + \cdots + b_m E_m$.

$$\mathcal{J}(\mathfrak{a}^\lambda) = \pi_* \mathcal{O}_X(K_{X/Y} - \lfloor \lambda D \rfloor) =$$

$$= \{h \in R \mid \nu_{E_i}(h) \geq \lfloor \lambda b_i \rfloor - a_i \ \ \forall i \ \text{ and } \ \nu_{\tilde{C}_i}(h) \geq \lfloor \lambda \rfloor\}$$

$$\Rightarrow \operatorname{lct}(C) = \min_{1 \leq j \leq m} \left\{ \bar{\alpha}_j := \frac{a_j + 1}{b_j} \right\}.$$

Important vertices (1)

- Suppose that

$$\underbrace{C_1, C_2, \dots, C_n}_{\text{Singular}}, \underbrace{C_{n+1}, C_{n+2}, \dots, C_r}_{\text{Smooth}}.$$

- $\mathcal{C}_i := \{P_j \in \mathcal{C} \mid \ \tilde{C}_i \text{ passes through } P_j\}, \quad 1 \leq i \leq r.$
- Terminal satellite point for $h \in R$: satellite point $P_j \in \mathcal{C}$ such that $\{P_k \in \mathcal{C} \setminus \{P_j\}$ s.t. the strict transform of H on X_k passes through P_k and $P_k \gtrsim P_j\}$ is either empty or its minimum (with respect to the ordering "infinitely near" $\gtrsim$) is a free point.

Important vertices (2)

$\mathcal{T}$

$$\mathcal{T} = \{P_{t_i} \mid i = 1, \ldots, n\},$$

where P_{t_i} := Minimum terminal satellite point of $\mathcal{C}_i$, $\quad 1 \leq i \leq n$.

$\mathcal{F}$

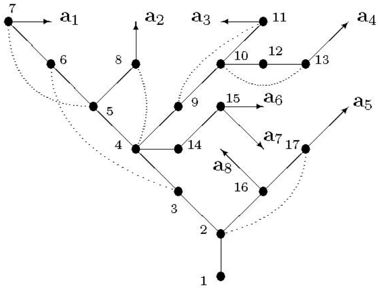
$$\mathcal{F} := \left\{P_j \in \mathcal{C} \mid P_{t_i} \gtrsim P_j \text{ for some } P_{t_i} \in \mathcal{T}\right\} \cup \mathcal{C}_{n+1} \cdots \cup \mathcal{C}_r.$$

Example: $f = f_1\ f_2\ f_3\ f_4\ f_5\ f_6\ f_7\ f_8$ (proximity graph)

$\mathcal{C}_1 = \{P_1, P_2, P_3, P_4, P_5, P_6, P_7\}.$
$\mathcal{C}_2 = \{P_1, P_2, P_3, P_4, P_5, P_8\}.$
$\mathcal{C}_3 = \{P_1, P_2, P_3, P_4, P_9, P_{10}, P_{11}\}.$
$\mathcal{C}_4 = \{P_1, P_2, P_3, P_4, P_9, P_{10}, P_{12}, P_{13}\}.$
$\mathcal{C}_5 = \{P_1, P_2, P_{16}, P_{17}\}.$
$\mathcal{C}_6 = \mathcal{C}_7 = \{P_1, P_2, P_3, P_4, P_{14}, P_{15}\}.$
$\mathcal{C}_8 = \{P_1, P_2, P_{16}\}.$

$$\mathcal{T} = \{P_{t_1} = P_7, P_{t_2} = P_8, P_{t_3} = P_{11}, P_{t_4} = P_{13}, P_{t_5} = P_{17}\}.$$

$\mathcal{F} = \mathcal{C}.$

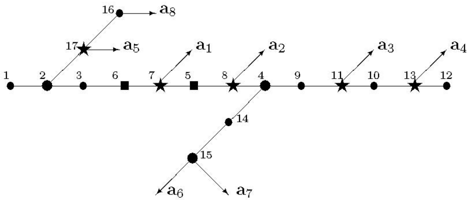


Example: $f = f_1\ f_2\ f_3\ f_4\ f_5\ f_6\ f_7\ f_8$ (dual graph)

$\mathcal{C}_1 = \{P_1, P_2, P_3, P_4, P_5, P_6, P_7\}$.
$\mathcal{C}_2 = \{P_1, P_2, P_3, P_4, P_5, P_8\}$.
$\mathcal{C}_3 = \{P_1, P_2, P_3, P_4, P_9, P_{10}, P_{11}\}$.
$\mathcal{C}_4 = \{P_1, P_2, P_3, P_4, P_9, P_{10}, P_{12}, P_{13}\}$.
$\mathcal{C}_5 = \{P_1, P_2, P_{16}, P_{17}\}$.
$\mathcal{C}_6 = \mathcal{C}_7 = \{P_1, P_2, P_3, P_4, P_{14}, P_{15}\}$.
$\mathcal{C}_8 = \{P_1, P_2, P_{16}\}$.

$$\mathcal{T} = \{P_{t_1} = P_7, P_{t_2} = P_8, P_{t_3} = P_{11}, P_{t_4} = P_{13}, P_{t_5} = P_{17}\}.$$

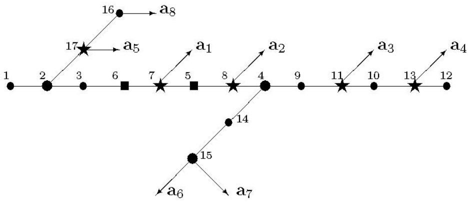
$\mathcal{F} = \mathcal{C}$.
Square: satellite point not in $\mathcal{T}$. Star: satellite point in $\mathcal{T}$.



Important vertices (3). Initial separating points

Set of initial separating points, $\mathcal{S}$:
Points $P_j \in \mathcal{C}$ such that two components C_{i_1} and C_{i_2} of C are freely separated at P_j. That is:

- $\max_{\gtrsim}(\mathcal{C}_{i_1} \cap \mathcal{C}_{i_2}) = P_j$. (*Separated* at P_j).
- $(f_{i_1} \mid f_{i_2}) = (0, c)$ -no common satellite points and c free ones- for some $c \leq \min\{l_0^{i_1}, l_0^{i_2}\}$. (*Freely separated* at P_j).



l_0^h (l_0^{h+1}): free points P_j through which the strict transform of H pass (and all satellite points P, satisfy $P \gtrsim P_j$) if H is not (is) singular.

$$\mathcal{S} = \{P_2, P_4, P_{15}\}.$$

$$\mathcal{V}_{\mathcal{T}} := \{\mathbf{v}_j | P_j \in \mathcal{T}\}, \quad \mathcal{V}_{\mathcal{S}} := \{\mathbf{v}_j | P_j \in \mathcal{S}\}$$

$$\mathcal{V} = \mathcal{V}_{\mathcal{T}} \cup \mathcal{V}_{\mathcal{S}}.$$

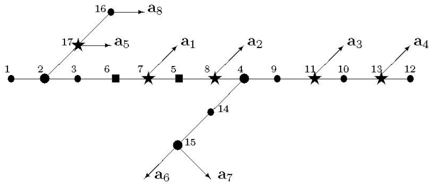
Left and right accessibility

In the dual graph:

$\leq$ denote the order induced by $\Gamma(C)$. $\mathbf{v}_{j_1} \leq \mathbf{v}_{j_2}$ means that $\mathbf{v}_{j_1}$ belongs to $[\mathbf{v}_1, \mathbf{v}_{j_2}]$. By convention, if $\mathbf{a}_i$ is an arrow that is a label of $\mathbf{v}_{j_2}$ then $\mathbf{v}_{j_1} \leq \mathbf{a}_i$ will mean $\mathbf{v}_{j_1} \leq \mathbf{v}_{j_2}$.

$$\mathbf{v}_j^{<} := \{\mathbf{a}_i \mid \mathbf{v}_j \not\leq \mathbf{a}_i\},$$

$$\mathbf{v}_j^{\geq} := \{\mathbf{a}_i \mid \mathbf{v}_j \leq \mathbf{a}_i\}.$$



$\sigma : \mathcal{V}_{\mathcal{F}}(:= \{\mathbf{v}_j | P_j \in \mathcal{F})\} \to \mathbb{Z}$ given by

$$\sigma(\mathbf{v}_j) = \sum_{\mathbf{a}_i \in \mathbf{v}_j^{<}} c_{ji} \bar{\beta}_0^i - \sum_{\mathbf{a}_i \in \mathbf{v}_j^{\geq}} \bar{\beta}_0^i,$$

$$c_{ji} := \begin{cases} \operatorname{card}([\mathbf{v}_1, \mathbf{a}_i] \cap [\mathbf{v}_1, \mathbf{v}_j] \cap \mathcal{V}_{\text{free}}) & \text{if } \operatorname{ter}([\mathbf{v}_1, \mathbf{a}_i] \cap [\mathbf{v}_1, \mathbf{v}_j]) \in \mathcal{S} \\ \bar{\beta}_1^i / \bar{\beta}_0^i & \text{otherwise.} \end{cases}$$

Theorem (I)

Let C be s above. Then:

- **(1)** There exists a vertex $\mathbf{v}_k \in \mathcal{V}$ satisfying the conditions:
  (a) $\sigma(\mathbf{v}_j) < 0$ for all $\mathbf{v}_j \in [\mathbf{v}_1, \mathbf{v}_k] \cap \mathcal{V}$ and
  (b) $\sigma(\mathbf{v}_j) \geq 0$ for all $\mathbf{v}_j \in \mathcal{V} \setminus [\mathbf{v}_1, \mathbf{v}_k]$.
- **(2)** The log-canonical threshold of C is the value $\overline{\alpha}_k$ above defined and it can be computed as follows:

Our result

Theorem (II)

- If $\mathbf{v}_k = \mathbf{v}_{t_i} \in \mathcal{V}_{\mathcal{T}}$, then

$$\overline{\alpha}_k = \overline{\alpha}_{t_i} = \frac{\bar{\beta}_0^i + \bar{\beta}_1^i}{\sum_{s=1}^r \delta_{is}},$$

where

$$\delta_{is} = \begin{cases} \bar{\beta}_0^i \bar{\beta}_1^s & \text{if either } s = i \text{, or } s \neq i \text{ and } \bar{\beta}_0^i \bar{\beta}_1^s = \bar{\beta}_1^i \bar{\beta}_0^s \leq I(f_i, f_s) \\ I(f_i, f_s) & \text{otherwise.} \end{cases}$$

- If $\mathbf{v}_k \in \mathcal{V}_{\mathcal{S}}$, then

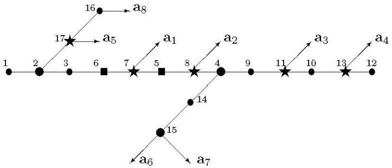
$$\overline{\alpha}_k = \frac{\bar{\beta}_0^{i_1} \bar{\beta}_0^{i_2} + I(f_{i_1}, f_{i_2})}{\bar{\beta}_0^{i_1} I(f_{i_1}, f_{i_2}) + \bar{\beta}_0^{i_2} \sum_{1 \leq s \leq r,\ s \neq i_1} I(f_{i_1}, f_s)},$$

where C_{i_1} and C_{i_2} are any two components which are freely separated at P_k.

Example (i)

$$\{\bar\beta_0^1, \bar\beta_1^1\} = \{5, 17\}$$
$$\{\bar\beta_0^2, \bar\beta_1^2\} = \{3, 11\}$$
$$\{\bar\beta_0^3, \bar\beta_1^3\} = \{2, 11, \}$$
$$\{\bar\beta_0^4, \bar\beta_1^4\} = \{2, 13\}$$
$$\{\bar\beta_0^5, \bar\beta_1^5\} = \{2, 5\}$$

$$\mathcal{V} = \mathcal{V}_{\mathcal{S}} \cup \mathcal{V}_{\mathcal{T}} = \{\mathbf{v}_2, \mathbf{v}_4, \mathbf{v}_7, \mathbf{v}_8, \mathbf{v}_{11}, \mathbf{v}_{13}, \mathbf{v}_{15}, \mathbf{v}_{17}\}.$$



$$\sigma(\mathbf{v}_2) = -\sum_{i=1}^{8} \bar\beta_0^i = -17.$$

$$\sigma(\mathbf{v}_7) = 2\bar\beta_0^5 + 2\bar\beta_0^8 - \bar\beta_0^1 - \bar\beta_0^2 - \bar\beta_0^3 - \bar\beta_0^4 - \bar\beta_0^6 - \bar\beta_0^7 = -4.$$

$$\sigma(\mathbf{v}_8) = 2\bar\beta_0^5 + 2\bar\beta_0^8 + \bar\beta_1^1 - \bar\beta_0^2 - \bar\beta_0^3 - \bar\beta_0^4 - \bar\beta_0^6 - \bar\beta_0^7 = 14.$$

Example (ii). Log-canonical threshold

Then, $\mathbf{v}_7$ is the distinguished vertex $\mathbf{v}_k$ and

$$\operatorname{lct}(C) = \overline{\alpha}_7 = \overline{\alpha}_{t_1} = \frac{\bar{\beta}_1^1 + \bar{\beta}_0^1}{\bar{\beta}_1^1 \bar{\beta}_0^1 + \sum_{s=2}^{8} I(f_1, f_s)} =$$

$$= \frac{17+5}{17\cdot 5 + 17\cdot 3 + 17\cdot 2 + 17\cdot 2 + 2\cdot 5\cdot 2 + 17\cdot 1 + 17\cdot 1 + 2\cdot 5\cdot 1}$$

$$= \frac{11}{134}.$$

Corollary: the case of 2 branches

Assume that the number of components of C is $r = 2$ and, without loss of generality, that $\bar{\beta}_1^1/\bar{\beta}_0^1 \leq \bar{\beta}_1^2/\bar{\beta}_0^2$. Then:

(a) If C_1 and C_2 are not freely separated, it holds that

$$\mathrm{lct}(C) = \begin{cases} \frac{\bar{\beta}_1^1+\bar{\beta}_0^1}{\bar{\beta}_1^1(\bar{\beta}_0^1+\bar{\beta}_0^2)} & \text{if } \bar{\beta}_1^1 \geq \bar{\beta}_0^2, \\ \frac{\bar{\beta}_1^2+\bar{\beta}_0^2}{\bar{\beta}_0^2(\bar{\beta}_1^1+\bar{\beta}_1^2)} & \text{otherwise.} \end{cases}$$

(b) If, on the contrary, C_1 and C_2 are freely separated,

$$\mathrm{lct}(C) = \begin{cases} \frac{\bar{\beta}_0^1\bar{\beta}_0^2+I(f_1,f_2)}{(\bar{\beta}_0^1+\bar{\beta}_0^2)I(f_1,f_2)} & \text{if } \frac{1}{c} \leq \frac{\bar{\beta}_0^2}{\bar{\beta}_0^1} \leq c, \\ \frac{\bar{\beta}_1^1+\bar{\beta}_0^1}{\bar{\beta}_0^1\bar{\beta}_1^1+I(f_1,f_2)} & \text{if } \frac{\bar{\beta}_0^2}{\bar{\beta}_0^1} < \frac{1}{c}, \\ \frac{\bar{\beta}_1^2+\bar{\beta}_0^2}{\bar{\beta}_0^2\bar{\beta}_1^2+I(f_1,f_2)} & \text{otherwise,} \end{cases}$$

c being the integer such that $(f_1 \mid f_2) = (0, c)$.

Lct of a complete ideal

Let $\mathfrak{a}$ be a complete ideal of finite co-length in *R* (local regular and bidimensional). $\mathfrak{a}$ has a unique factorization $\mathfrak{a} = \mathfrak{p}_1^{n_1} \cdots \mathfrak{p}_r^{n_r}$ as a product of simple complete ideals. Then,

$$\mathrm{lct}(\mathfrak{a}) = \mathrm{lct}(\sum_{i=1}^{r} D_i)$$

where, for each $i = 1, \ldots, r$, D_i is a sum of n_i suitable chosen *general* curves of the ideal $\mathfrak{p}_i$.

Suitable chosen means that the curves meet the corresponding divisor at different points.

Complete Ideal

THANK YOU FOR YOUR ATTENTION

ORGANIZERS: THANK YOU VERY MUCH