Castelnuovo-Mumford regularity of projective monomial curves associated to arithmetic sequences

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Joint work with I. Bermejo

Definitions and notations

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Let K be an algebraically closed field and $\boldsymbol{R = K[x_0, \ldots, x_{n+1}]}$ and $K[s,t]$ two polynomial rings over K.

Let $m_0 < \ldots < m_n$ be an **arithmetic sequence** of positive integers, i.e., $m_i = m_0 + id$, with $d \in \mathbb{Z}^+$, and assume that $\gcd\{m_0, d\} = 1$.

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Consider the **projective monomial curve** $\mathcal{C} \subset \mathbb{P}_K^{n+1}$ parametrically defined by

$$x_0 = t^{m_0} s^{m_n - m_0}, \ldots, x_{n-1} = t^{m_{n-1}} s^{m_n - m_{n-1}}, x_n = t^{m_n}, x_{n+1} = s^{m_n}.$$

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Consider the k-algebra homomorphism $\varphi : R \to K[t,s]$ induced by $\varphi(x_i) = t^{m_i} s^{m_n - m_i}$ for $i \in \{0, \ldots, n-1\}$, $\varphi(x_n) = t^{m_n}$ and $\varphi(x_{n+1}) = s^{m_n}$. The binomial prime ideal $\mathcal{P} := ker(\varphi) \subset R$ is the **defining ideal of the projective monomial curve** $\mathcal{C}$.

Definitions and notations

The **Castelnuovo-Mumford regularity** or simply **regularity reg($\mathcal{C}$)** of $\mathcal{C}$ is, by definition, the Castelnuovo-Mumford regularity **reg($\mathcal{P}$)** of $\mathcal{P}$, and can be defined as follows: if

$$0 \to \bigoplus_{j=1}^{\beta_p} R(-e_{pj}) \xrightarrow{\varphi_p} \cdots \xrightarrow{\varphi_1} \bigoplus_{j=1}^{\beta_0} R(-e_{0j}) \xrightarrow{\varphi_0} \mathcal{P} \to 0$$

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is a minimal graded free resolution of $\mathcal{P}$, then

$$\mathbf{reg}(\mathcal{P}) = \max\{e_{ij} - i;\ 0 \le i \le p,\ 1 \le j \le \beta_i\}$$

AIM OF THE WORK

The **aim** of this work consists of **obtaining an explicit formula for $\operatorname{reg}(\mathcal{C})$ in terms of the arithmetic sequence $\mathbf{m_0}, \ldots, \mathbf{m_n}$** that leads us to give an efficient algorithm for computing $\operatorname{reg}(\mathcal{C})$.

Motivation

The **Castelnuovo-Mumford regularity**

- is one of the most important invariant of a graded module.
- is a good measure of the complexity of computing Gröbner bases.

Outline

Bermejo-Gimenez introduced a class of monomial ideals called **monomial ideals of nested type**,

Definition

A monomial ideal $I \subset R$ is said to be of **nested type** if, for any prime ideal $\mathfrak{p}$ associated to I, there exists $i \in \{0, \ldots, n+1\}$ such that $\mathfrak{p} = (x_0, \ldots, x_i)$.

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These monomial ideals have nice combinatorial properties that make the computation of its regularity easy. In particular, the regularity of a monomial ideal of nested type can be easily read from its irredundant irreducible decomposition.

An **irreducible decomposition** of a monomial ideal I is an expression of I as intersection of irreducible monomial ideals. If there is no redundant ideal in this intersection, this expression is unique and we call it **the irredundant irreducible decomposition** of I.

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Theorem (Bermejo-Gimenez)

Let $I \subset R$ be a monomial ideal of nested type. Then

$$\mathrm{reg}(I) = \max\{\mathrm{reg}(\mathfrak{q}_i);\ 1 \leq i \leq r\},$$

where $I = \mathfrak{q}_1 \cap \ldots \cap \mathfrak{q}_r$ is the irredundant irreducible decomposition of I.

Theorem (Bayer-Stillman, Bermejo-Gimenez)

Let $I \subset R$ be a homogeneous ideal such that its initial ideal $\mathrm{in}(I)$ with respect to the reverse lexicographic order with $x_0 > \ldots > x_{n+1}$ is of nested type. Then,

$$\mathrm{reg}(I) = \mathrm{reg}(\mathrm{in}(I))$$

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The **initial ideal w.r.t. the reverse lexicographic order of the defining ideal of a projective monomial curve is a monomial ideal of nested type**.

Lemma (α, k)

Let $m_0 < \ldots < m_n$ be an arithmetic sequence with $\gcd\{m_0, d\} = 1$. If we set

$$v := \min\{b \in \mathbb{Z}^+ \;/\; b\, m_n \in \mathbb{N}\, m_0 + \cdots + \mathbb{N}\, m_{n-1}\},$$

then there exits a unique pair of integers (α, k) such that

$$v\, m_n = \alpha\, m_0 + m_k,$$

where $\alpha > 0$ and k : $\max\{0, n - m_0\} \leq k \leq n - 1$.

Example

Set $m_0 = 8$, $m_1 = 15$, $m_2 = 22$, $m_3 = 29$, $m_4 = 36$, $m_5 = 43$.
We have:

- $m_5 = 43 \notin \mathbb{N}\{m_0, \dots, m_4\}$
- $2m_5 = 86 = 8 \cdot 8 + 22 = 8m_0 + m_2$

Then $v = 2$, $\alpha = 8$ and $k = 2$.

Theorem

The set

$$G = \{x_i x_j - x_{i-1}x_{j+1} \;/\; 1 \le i \le j \le n-1\}$$
$$\cup \{x_0^{\alpha} x_{k-i} - x_{n-i} x_n^{v-1} x_{n+1}^{\alpha+1-v} \;/\; 0 \le i \le k\}$$

is a minimal Gröbner basis of $\mathcal{P}$ with respect to the reverse lexicographic order.

Corollary

$$\mathrm{in}(\mathcal{P}) = \langle\{\mathbf{x_i x_j};\ 1 \leq i \leq j \leq n-1\}\rangle + \langle\{\mathbf{x_0^{\alpha} x_i};\ 0 \leq i \leq k\}\rangle$$

Corollary

$$\mathrm{in}(\mathcal{P}) = \langle\{\mathbf{x_i x_j};\ 1 \leq i \leq j \leq n-1\}\rangle + \langle\{\mathbf{x_0^{\alpha} x_i};\ 0 \leq i \leq k\}\rangle$$

Corollary

$\mathcal{C}$ is an arithmetically Cohen-Macaulay curve.

Proposition

1. *If* $\boldsymbol{k} < \boldsymbol{n} - \mathbf{1}$*, then*

$$\mathrm{in}(\mathcal{P}) = \bigcap_{1 \le i \le n-1} \langle x_0^{\delta_i}, x_1, \ldots, x_{i-1}, x_i^2, x_{i+1}, \ldots, x_{n-1} \rangle$$

is the irredundant irreducible decomposition of $\mathrm{in}(\mathcal{P})$*,*
where $\delta_i = \begin{cases} \alpha, & \text{if } 1 \le i \le k \\ \alpha + 1, & \text{if } k < i \le n-1, \end{cases}$

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is the irredundant irreducible decomposition of $\mathrm{in}(\mathcal{P})$, *where* $\delta_i = \begin{cases} \alpha, & \text{if } 1 \leq i \leq k \\ \alpha + 1, & \text{if } k < i \leq n-1, \end{cases}$

2. *If* $\mathbf{k} = \mathbf{n} - \mathbf{1}$, *then*

$$\mathrm{in}(\mathcal{P}) = \left[\bigcap_{1 \leq i \leq n-1} \langle x_0^{\alpha}, x_1, \ldots, x_{i-1}, x_i^2, x_{i+1}, \ldots, x_{n-1} \rangle \right] \bigcap \langle x_0^{\alpha+1}, x_1, \ldots, x_{n-1} \rangle$$

is the irredundant irreducible decomposition of $\mathrm{in}(\mathcal{P})$.

Lemma

Let $\mathfrak{q} = \langle x_{i_1}^{a_{i_1}}, \ldots, x_{i_s}^{a_{i_s}} \rangle \subset R$ *be an irreducible monomial ideal, where* $a_{i_j} \in \mathbb{Z}^+$ *for all* $j \in \{1, \ldots, s\}$*, where* $\{i_1, \ldots, i_s\} \subseteq \{0, \ldots, n+1\}$*. Then*

$$\mathrm{reg}(\mathfrak{q}) = \sum_{j=0}^{s} (a_{i_j} - 1) + 1$$

Theorem

Let $m_0 < \cdots < m_n$ be an arithmetic sequence with $\gcd\{m_0, d\} = 1$. If (α, k) is the pair of integers of Lemma (α, k), then

$$\operatorname{reg}(\mathcal{C}) = \begin{cases} \alpha + 2, & \text{if } k < n - 1 \\ \alpha + 1, & \text{if } k = n - 1 \end{cases}$$

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Example

In the previous example, we obtained that $v = 2$, $\alpha = 8$ and $k = 2$, thus

$$\mathrm{reg}(\mathcal{C}) = \alpha + 2 = 8 + 2 = 10$$

This is because $k = 2 < 4 = n - 1$.

INPUT: $m_0 < \cdots < m_n$

OUTPUT: **reg**$(\mathcal{C})$

$v := \min\{b \in \mathbb{Z}^+ \;/\; bm_n \in \mathbb{N}\{m_0, \ldots, m_{n-1}\}\}$
$k := n - 1$
WHILE $m_0 \nmid (v\, m_n - m_k)$
    $k := k - 1$
$\alpha := (v\, m_n - m_k)/m_0$
IF $k < n - 1$
    RETURN $\alpha + 2$
RETURN $\alpha + 1$

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