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Polar quotients and singularities at infinity of polynomials in two complex variables

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Abstract. Using the notion of the maximal polar quotient we characterize the critical values at infinity of polynomials in two complex variables. As an application we give a necessary and sufficient condition for a family of affine plane curves to be equisingular at infinity.

Introduction. For any reduced projective curve $C \subset \mathbb{P}^2(\mathbb{C})$ given by a homogenuous equation $F = 0$ and for every point $p = (a_0 : a_1 : a_2)$ not lying on C we consider the polar curve $\nabla_p C$ (possibly with multiple components) given by the equation

$$\nabla_p C = a_0 \frac{\partial F}{\partial X_0} + a_1 \frac{\partial F}{\partial X_1} + a_2 \frac{\partial F}{\partial X_2} = 0.$$

Let $L \subset \mathbb{P}^2(\mathbb{C})$ be a line. Suppose that $L \not\subset C$ and fix a point $p \in L \setminus C$. Then L is not a component of $\nabla_p C$ and we may consider the polar quotients of C with respect to L at a point $o \in C \cap L$:

$$\frac{\operatorname{ord}_\gamma C}{\operatorname{ord}_\gamma L}, \qquad \gamma \text{ runs over branches of } \nabla_p C \text{ with center at } o.$$

The polar quotients do not depend on the choice of $p \in L \setminus C$. They are identical with the local polar quotients of the germ (C, o) with respect to (L, o) ([LMW], [Pł2]).

Let

$$q_o(C, L) = \sup\{q : q \text{ is a polar quotient of } C \text{ with respect to } L \text{ at } o\}.$$

Clearly the set of polar quotients is empty if and only if o is reqular on C and L is not tangent to C at o. In this situation $q_o(C, L) = -\infty$ for by convention $\sup \emptyset = -\infty$. Let $\mu_o(C)$ be the Milnor number of C at o. If

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$q_o(C, L) \neq -\infty$ then we have

$$\frac{\mu_o(C)}{(C \cdot L)_o - 1} + 1 \leq q_o(C, L) \leq \mu_o(C) + 1$$

where $(C \cdot L)_o$ is the intersection number of C and L at o. Note that $q_o(C, L) = \mu_o(C) + 1$ provided $(C \cdot L)_o = 2$. We omit the easy proof of the above estimate (for the right inequality see [Te2], Remark on p. 202).

The maximal polar quotient $q_o(C, L)$ is a topological invariant of the germ $(C \cup L, o)$. It can be calculated explicitly in terms of the characteristics of the branches and their intersection multiplicities ([Pł2], Theorem 1.3).

Let C be a reduced projective curve and L a line such that $L \not\subset C$. Then we put

$$q(C, L) = \sup\{q_o(C, L) : o \in C \cap L\}.$$

Hence $q(C, L) = -\infty$ if and only if C and L meet with multiplicity 1. If $q(C, L) \neq -\infty$ then $q(C, L)$ is a rational number and

$$1 \leq q(C, L) \leq (d - 1)^2 + 1$$

where $d = \deg C$.

Now, let us consider a reduced polynomial $f = f(X, Y) \in \mathbb{C}[X, Y]$ (i.e. without multiple factors) of degree $d > 0$ and let C be the projective closure of the affine curve $f(X, Y) = 0$. Let $\mathbb{P}^2(\mathbb{C}) = \mathbb{C}^2 \cup \mathbb{L}_\infty$ where $\mathbb{L}_\infty$ is the line at infinity. We call $q(C, \mathbb{L}_\infty)$ the *maximal polar quotient at infinity* of the affine curve $f(X, Y) = 0$. If C is the projective closure of an affine curve with multiple components then we put $q(C, \mathbb{L}_\infty) = +\infty$. Throughout this paper we use the usual conventions on the symbols $-\infty$ and $+\infty$.

Our purpose is to calculate the maximal polar quotient of the curves $f(X, Y) - t = 0$, $t \in \mathbb{C}$, in terms of the discriminant. We use this notion to characterize the critical values at infinity. The proof of our main result is based on the notion of the Łojasiewicz exponent at infinity (see [CK1], [CK2], [CN-H], [H], [Pł1]).

1. Result. Let $f = f(X, Y) \in \mathbb{C}[X, Y]$ be a reduced polynomial of degree $d > 0$ and let $F = F(X, Y, Z) \in \mathbb{C}[X, Y, Z]$ be the homogeneous form corresponding to f. Let $C_\infty = C \cap \mathbb{L}_\infty$ be the set of points at infinity of the projective curve C given by the equation $F(X, Y, Z) = 0$.

We consider the projective curves C^t (possibly with multiple factors) given by the equations

$$F(X, Y, Z) - tZ^d = 0.$$

Clearly $C^t \cap \mathbb{L}_\infty = C_\infty$. Let $\mu_p^t = \mu_p(C^t)$ be the Milnor number of C^t at $p \in C_\infty$. If a multiple component of C^t passes through p then $\mu_p^t = +\infty$.

Let $\mu_p^{\min} = \inf\{\mu_p^t : t \in \mathbb{C}\}$ and

$$\Lambda(f) = \{t \in \mathbb{C} : \text{there is a } p \in C_\infty \text{ such that } \mu_p^t > \mu_p^{\min}\}.$$

The set $\Lambda(f)$ is finite (see [B]). The elements of $\Lambda(f)$ are called *critical values at infinity* of the polynomial $f : \mathbb{C}^2 \to \mathbb{C}$. If C^t has a multiple component then $t \in \Lambda(f)$. Different definitions of critical values at infinity are discussed in [D]. The elements of $\mathbb{C} \setminus \Lambda(f)$ are called *regular values at infinity*.

For every $t \in \mathbb{C}$ we put

$$\lambda^t(f) = \sum_{p \in C_\infty} (\mu_p^t - \mu_p^{\min})$$

if C^t is reduced and $\lambda^t(f) = +\infty$ if not. Note that if C and $\mathbb{L}_\infty$ meet with multiplicity 1 then $\Lambda(f) = \emptyset$ and $q(C^t, \mathbb{L}_\infty) = -\infty$ for all $t \in \mathbb{C}$. The following assumptions will be made throughout this paper:

- f is a reduced polynomial of degree $d > 0$,
- the projective closure C of the affine curve $f(X, Y) = 0$ and the line at infinity $\mathbb{L}_\infty$ meet with multiplicity > 1 at a point of $C \cap \mathbb{L}_\infty$.

Now suppose additionally that $\deg_Y f = \deg f = d > 1$ and let T be a new variable. Consider the Y-discriminant $\Delta(X, T) = \operatorname{disc}_Y(f(X, Y) - T)$ of the polynomial $f(X, Y) - T \in \mathbb{C}[X, T][Y]$. Obviously $\Delta(X, t) \neq 0$ in $\mathbb{C}[X]$ if and only if the polynomial $f(X, Y) - t$ is reduced. Write

$$\Delta(X, T) = \Delta_0(T) X^N + \ldots + \Delta_N(T) \quad \text{with } \Delta_0(T) \neq 0.$$

The following proposition is well known.

Proposition 1.1.

$$\Lambda(f) = \{t \in \mathbb{C} : \Delta_0(t) = 0\}, \qquad \lambda^t(f) = N - \deg_X \Delta(X, t).$$

A simple, intersection-theoretic proof of Proposition 1.1 is given in [K] (see also [GP2]).

Our main result is the following.

Theorem 1.2. *With the notation introduced above*,

$$q(C^t, \mathbb{L}_\infty) = \begin{cases} d - \left(\max\limits_{i=1}^{N} \dfrac{\deg \Delta_i}{i} \right)^{-1} & \textit{if } \Lambda(f) = \emptyset, \\ d & \textit{if } \Lambda(f) \neq \emptyset \textit{ and } t \notin \Lambda(f), \\ d + \left(\min\limits_{i=0}^{\lambda^t(f)-1} \dfrac{\operatorname{ord}_t \Delta_i}{\lambda^t(f) - i} \right)^{-1} & \textit{if } \Lambda(f) \neq \emptyset,\ t \in \Lambda(f) \textit{ and} \\ & C^t \textit{ is reduced.} \end{cases}$$

Recall that according to our convention $q(C^t, \mathbb{L}_\infty) = +\infty$ if C^t is not reduced. The proof of Theorem 1.2 is given in Section 3. Now, we present some applications.

Corollary 1.3. (i) *If* $\Lambda(f) = \emptyset$ *then* $q(C^t, \mathbb{L}_\infty) = q(C, \mathbb{L}_\infty) < d$ *for all* $t \in \mathbb{C}$.

(ii) *If* $\Lambda(f) \neq \emptyset$ *then* $q(C^t, \mathbb{L}_\infty) = d$ *if* $t \in \mathbb{C} \setminus \Lambda(f)$ *and* $q(C^t, \mathbb{L}_\infty) > d$ *if* $t \in \Lambda(f)$.

The above corollary shows that $q(C^t, \mathbb{L}_\infty)$, like $\lambda^t(f)$, is an upper semicontinuous function of t. This contrasts with the fact that the maximal polar quotient is neither upper nor lower semicontinuous in the family of hypersurfaces (see [Te2], p. 201).

Remark 1.4. If $\Lambda(f) \neq \emptyset$ then $d \leq q(C^t, \mathbb{L}_\infty) \leq d + \lambda^t(f)$ for all $t \in \mathbb{C}$.

The characterization of regular values at infinity presented below is due to Neumann and Lê Văn Thàn who used topological methods.

Corollary 1.5 (see [Lê] and [NL]). *A number* $t_0 \in \mathbb{C}$ *is a regular value at infinity of* f *if and only if* $q(C^{t_0}, \mathbb{L}_\infty) \leq d$.

The family of polynomials $f(X,Y) - t$, $t \in \mathbb{C}$, is said to be *equisingular at infinity* if $\Lambda(f) = \emptyset$. All polynomials of an equisingular family are reduced. Theorem 1.2 implies the following criterion of equisingularity.

Corollary 1.6. *Let* $f = f(X,Y) \in \mathbb{C}[X,Y]$ *be a reduced polynomial of degree* $d > 1$. *Then the following two conditions are equivalent*:

(i) *the family* $f(X,Y) - t$ *is equisingular*,
(ii) $q(C, \mathbb{L}_\infty) < d$.

To give an application of Corollary 1.6 let us recall

The Abhyankar–Moh inequality. *Let* C *be a reduced projective plane curve of degree* $d > 1$. *Suppose that* $o \in C$ *is a unibranch point of* C (*i.e. only one branch of* C *passes through* o) *such that the unique tangent* L *to* C *at* o *does not intersect* C *at points different from* o. *Then* $q_o(C, L) < d$.

The original Abhyankar–Moh inequality is stated in terms of Puiseux expansions [AM]. The formulation above is given in [GP1] (Theorem 2.2). The Abhyankar–Moh inequality and Corollary 1.6 imply

Corollary 1.7 (the Ephraim equisingularity theorem, [Eph], Theorem 3.4). *Let* $f = f(X,Y)$ *be a reduced polynomial such that the projective closure* C *of the affine curve* $f(X,Y) = 0$ *has only one branch at infinity. Then the family* $f(X,Y) - t$ *is equisingular.*

Here is another application of Corollary 1.6:

Corollary 1.8. *Let* $f = f(X,Y)$ *be a reduced polynomial of degree* $d > 1$ *with* $d - 1$ *points at infinity. Let* $o \in C_\infty$ *be the unique point such that* $(C \cdot \mathbb{L}_\infty)_o = 2$. *Then the family* $f(X,Y) - t$ *is equisingular if and only if the Milnor number* $\mu_o = \mu_o(C)$ *is strictly less than* $d - 1$.

To prove Corollary 1.8 it suffices to recall that the condition $(C\cdot\mathbb{L}_\infty)_o = 2$ implies $q_o(C, \mathbb{L}_\infty) = \mu_o + 1$ and apply Corollary 1.6.

2. Proof. The proof of our main result is based on some properties of the Łojasiewicz exponents. Let f, h be two polynomials of two complex variables. We assume that $h \not\equiv$ const. The *Łojasiewicz exponent at infinity* $\mathcal{L}_\infty(f|h)$ of the polynomial function $f|h^{-1}(0)$ is, by definition, the least upper bound of the set

$$\{\Theta \in \mathbb{R} : \exists C, R > 0 \ \forall z \in h^{-1}(0) \ (|z| \geq R \Rightarrow |f(z)| \geq C|z|^{\Theta})\}.$$

Then $\mathcal{L}_\infty(f|h) > -\infty$ if and only if the set

$$\{(x, y) \in \mathbb{C}^2 : f(x, y) = h(x, y) = 0\}$$

is finite. To calculate $\mathcal{L}_\infty(f|h)$ we use meromorphic parametrizations $p(T) = (x(T), y(T))$ where $x(T), y(T) \in \mathbb{C}((T))$ are Laurent series convergent in a punctured disc at the origin such that $\min\{\operatorname{ord} x(T), \operatorname{ord} y(T)\} < 0$. In what follows we put $\operatorname{ord} p(T) = \min\{\operatorname{ord} x(T), \operatorname{ord} y(T)\}$.

LEMMA 2.1.

$$\mathcal{L}_\infty(f|h) = \inf\left\{\frac{\operatorname{ord}(f\circ p)}{\operatorname{ord} p} : p \textit{ is a meromorphic parametrization such that } h\circ p \equiv 0\right\}.$$

Proof. Let Γ be the image of the set $\Delta \setminus \{0\}$, where Δ is a small disc around $0 \in \mathbb{C}$, under the mapping $\Delta \setminus \{0\} \ni t \mapsto p(t) \in \mathbb{C}$. One easily sees that $\Theta_\Gamma = \operatorname{ord}(f\circ p)/\operatorname{ord} p$ is the greatest number Θ such that

$$|f(z)| \geq \text{const}\cdot|z|^{\Theta} \quad \text{for } z \in \Gamma \text{ and } |z| \to +\infty.$$

Let $V = \{(x, y) \in \mathbb{C}^2 : h(x, y) = 0\}$. Then there exists a neighbourhood of infinity $\omega \subset \mathbb{C}^2$ (i.e. $\mathbb{C}^2 \setminus \omega$ is compact) and a finite sequence of meromorphic parametrizations p_i $(i = 1, \ldots, r)$ such that

$$V \cap \omega = \bigcup_{i=1}^{r} p_i(\Delta \setminus \{0\}) \quad \text{(disjoint union, see [CK1]).}$$

Therefore

$$\mathcal{L}_\infty(f|h) = \min_{i=1}^{n}\{\Theta_{\Gamma_i}\}, \quad \text{where} \quad \Gamma_i = p_i(\Delta \setminus \{0\}),$$

and the lemma follows.

A branch $\gamma \subset \mathbb{P}^2(\mathbb{C}) = \mathbb{C}^2 \cup \mathbb{L}_\infty$ is a branch at infinity if the centre of γ is on $\mathbb{L}_\infty$ and $\gamma \not\subset \mathbb{L}_\infty$. We say that a meromorphic parametrization $p(T) = (x(T), y(T))$ is a (meromorphic) *parametrization of a branch γ at infinity* if γ is given in projective coordinates by $(T^k x(T) : T^k y(T) : T^k)$ where

$k = -\operatorname{ord} p$. Any branch at infinity has a meromorphic parametrization. We check easily that $\operatorname{ord} p = -\operatorname{ord}_\gamma \mathbb{L}_\infty$ and

$$\operatorname{ord}(f \circ p) = \operatorname{ord}_\gamma \left(\frac{F}{Z^d}\right) = \operatorname{ord}_\gamma F - d \operatorname{ord}_\gamma Z$$

where $d = \deg f$ and F is the homogeneous form determined by f.

Hence we get

PROPOSITION 2.2. *With the notation introduced above*

$$\mathcal{L}_\infty(f|h) = d - \sup\{\operatorname{ord}_\gamma F / \operatorname{ord}_\gamma Z : \gamma \textit{ is a branch at infinity of the projective closure of } h = 0\}.$$

For every nonzero polynomial

$$R(X,T) = R_0(T)X^N + \ldots + R_N(T), \qquad R_0(T) \neq 0,$$

we define the *Puiseux exponent* $\pi(R)$ of R by putting

$$\pi(R) = \begin{cases} \max\limits_{j=1}^{N} \left(\dfrac{\deg R_j}{j}\right) & \text{if } R_0(T) \equiv \text{const}, \\ \infty & \text{if } R_0(0) \neq 0 \text{ and } R_0(T) \not\equiv \text{const}, \\ -\min\limits_{j=1}^{r} \left(\dfrac{\operatorname{ord}_0 R_j}{r+1-j}\right) & \text{if } R_0(0) = \ldots = R_r(0) = 0 \\ & \quad \text{and } R_{r+1}(0) \neq 0, \\ 0 & \text{if } R(X,0) \equiv 0. \end{cases}$$

We set $1/\infty = 0$.

LEMMA 2.3. *Suppose that* $R(X,0) \not\equiv 0$.

(i) *There exist positive constants* A, B *such that*

$$\{(x,t) : |x| > B \textit{ and } R(x,t) = 0\} \subset \{(x,t) : |x| > B \textit{ and } A|x|^{1/\pi(R)} \leq |t|\},$$

(ii) *If*

$$\{(x,t) : |x| > B \textit{ and } R(x,t) = 0\} \subset \{(x,t) : |x| > B \textit{ and } A|x|^{\nu} \leq |t|\}$$

for some $A, B > 0$ *and* $\nu \in \mathbb{R}$ *then* $\nu \leq 1/\pi(R)$.

Proof. If $R_0(T) \equiv \text{const}$ then Lemma 2.3 reduces to Lemma 2.1 of [Pł1]. If $R_0(T) \not\equiv \text{const}$ then the lemma follows from Lemmas 8.2 and 8.3 of [CK2].

The following proposition is a version of a result due to Chądzyński and Krasiński [CK2, Theorems 3.1–3.3].

PROPOSITION 2.4. *Let* $\deg_Y h = \deg h > 0$. *Consider*

$$R(X,T) = \operatorname{res}_Y(h(X,Y),\, f(X,Y) - T),$$

the Y-resultant of the polynomials $h(X,Y), f(X,Y)-T \in \mathbb{C}[X,T][Y]$. Assume $R(X,0)\neq 0$ in $\mathbb{C}[X]$. Then

$$\mathcal{L}_\infty(f|h) = \frac{1}{\pi(R)}.$$

Proof. Since $\deg_Y h = \deg h > 0$ we get

$$\{(x,y) : |x| > B \text{ and } h(x,y)=0\} \subset \{(x,y) : |x| > B \text{ and } |y| \le D|x|\}$$

for some constants $B, D > 0$. Therefore on the set $\{(x,y) : h(x,y)=0,\ |x|>B\}$ the inequality $|f(z)| \ge |z|^\Theta$ is equivalent to $|f(z)| \ge C_1|x|^\Theta$ with $C_1 > 0$ dependent on C and D.

Now fix $\Theta \in \mathbb{R}$. By the definition of the resultant the following two conditions are equivalent:

(1) $\{(x,y)\in\mathbb{C}^2 : h(x,y)=0,\ |x|>B\} \subset \{(x,y)\in\mathbb{C}^2 : |f(x,y)| \le C_1|x|^\Theta\}$,

(2) $\{(x,t)\in\mathbb{C}^2 : R(x,t)=0,\ |x|>B\} \subset \{(x,t)\in\mathbb{C}^2 : |t| \le C_1|x|^\Theta\}$.

Proposition 2.4 now follows from Lemma 2.3.

Remark 2.5. The Puiseux exponent $\pi(R)$ can be interpreted in terms of the Newton diagram of the polynomial $R(X,T)$ (see [GP2], Proposition 2.8 and remark to Proposition 2.8).

Remark 2.6. Let $R^t(X,T) = R(X,T+t)$ for $t\in\mathbb{C}$. Then $\pi(R^t)=\pi(R)$ if $R_0(T)\equiv \text{const}$, $\pi(R^t)=\infty$ if $R_0(t)\neq 0$ and $R(T)\not\equiv\text{const}$, and

$$\pi(R^t) = -\min_{j=0}^{r(t)}\left(\frac{\operatorname{ord}_t R_j}{r(t)+1-j}\right)$$

if $R_0(t) = \ldots = R_{r(t)}(t) = 0$ and $R_{r(t)+1}(t)\neq 0$. Note that $r(t)+1 = \deg_X R(X,T) - \deg R(X,t)$.

Now we can prove our main result.

Proof of Theorem 1.2. Assume that $\deg_Y f = \deg f = d > 1$ and put

$$C = \left(\text{projective closure of the curve } \left\{\frac{\partial f}{\partial Y}=0\right\}\right) = \left\{\frac{\partial F}{\partial Y}=0\right\}$$

where F is the homogeneous form corresponding to f. By Proposition 2.2 and definition of $q(C,\mathbb{L}_\infty)$ we get

$$\begin{aligned}\mathcal{L}_\infty\left(f\,\middle|\,\frac{\partial f}{\partial Y}\right) &= d - \sup\{\operatorname{ord}_\gamma F/\operatorname{ord}_\gamma Z : \gamma \text{ is a branch at infinity} \\ &\qquad\qquad \text{of the polar } \partial F/\partial Y = 0\} \\ &= d - q(C,\mathbb{L}_\infty).\end{aligned}$$

Applying the formula $\mathcal{L}_\infty(f|\partial f/\partial Y) = d - q(C,\mathbb{L}_\infty)$ to the polynomials $f-t$, $t\in\mathbb{C}$, we get

$$q(C^t,\mathbb{L}_\infty) = d - \mathcal{L}_\infty\left(f-t\,\middle|\,\frac{\partial f}{\partial Y}\right). \tag{1}$$

Let $\Delta(X,T) = \Delta_0(T)X^N + \ldots + \Delta_N(T)$, $\Delta_0(T)\not\equiv 0$, be the Y-discriminant of $f(X,Y)-T$, i.e. the Y-resultant of $f(X,Y)-T$ and $(\partial f/\partial Y)(X,Y)$. Let $\Delta^t(X) = \Delta(X,t)$. Consequently, $\deg\Delta^t(X) = \deg_X\Delta(X,T) - \lambda^t(f)$ by Proposition 1.1. Using Proposition 2.4 we get

$$\mathcal{L}_\infty\left(f-t\,\middle|\,\frac{\partial f}{\partial Y}\right) = \frac{1}{\pi(\Delta^t)} \tag{2}$$

provided that $f-t$ is reduced. Now, Theorem 1.2 follows from (1), (2) and from Remark 2.6.

3. Uniform estimation of polar quotients. It is natural to ask how large $q_o(C,L)$ can be when $\deg C = d$ is given. For every $d>1$ we put

$$q(d) = \sup\{q_o(C,L) : \exists C,L\ \deg C = d,\ o\in C\cap L \text{ and } L\not\subset C\}.$$

In the definition of $q(d)$ one may restrict oneself to the pairs C,L where C and L are transverse (see [Pł2], Corollary 1.4).

One easily sees that $q_o(C,L) = b/a$ where $a,b>0$ are integers and $a\le\deg C-1$. Therefore the set of all polar quotients $q_o(C,L)$ with $\deg C = d$ is finite and we get

Property 3.1. *The number $q(d)$ is rational.*

Recall that a curve C has an *A_k-singularity* at $o\in C$ if $\operatorname{ord}_o C = 2$ and $\mu_o(C) = k$. Let

$$k(d) = \max\{k : \text{there is a curve } C \text{ of degree } d \text{ with } A_k\text{-singularity}\}.$$

Property 3.2. $k(d)+1\le q(d)\le (d-1)^2+1$.

Proof. If C is of degree d with $A_{k(d)}$-singularity at $o\in C$ then $\mu_o(C) = k(d)$, $\operatorname{ord}_o C = 2$ and the polar quotient of C at o with respect to a transverse line L equals $\mu_o(C)+1 = k(d)+1$. Thus $k(d)+1\le q(d)$.

The second estimate follows from the inequalities

$$q_o(C,L)\le\mu_o(C)+1\le(d-1)^2+1.$$

If f is a polynomial with an isolated singularity at $0\in\mathbb{C}^2$ then its *Łojasiewicz exponent* $\mathcal{L}_0(f)$ is defined to be

$$\mathcal{L}_0(f) = \inf\{\Theta>0 : |\operatorname{grad} f(z)|\ge\text{const}\cdot|z|^\Theta \text{ for small } |z|\}.$$

According to Teissier [Te1], $\mathcal{L}_0(f)+1 =$ the maximal polar quotient of the curve $f=0$ at $0\in\mathbb{C}^2$ with respect to a generic line. Thus we get

PROPERTY 3.3. *If f is a polynomial of degree $d > 1$ with an isolated singularity at $0 \in \mathbb{C}^2$ then $\mathcal{L}_0(f) \leq q(d) - 1$.*

The above estimate is optimal: for every integer $d > 1$ there is a polynomial of degree d such that $\mathcal{L}_0(f) = q(d) - 1$. If $f : \mathbb{C}^2 \to \mathbb{C}$ is a polynomial with isolated critical points then the Łojasiewicz exponent at infinity $\mathcal{L}_\infty(f)$ is defined to be

$$\mathcal{L}_\infty(f) = \sup\{\Theta \in \mathbb{R} : |\operatorname{grad} f(z)| \geq \text{const} \cdot |z|^\Theta \text{ for large } |z|\}.$$

The following result is a reformulation of Proposition 6 from [CN-H]:

PROPOSITION 3.4. *Let $f : \mathbb{C}^2 \to \mathbb{C}$ be a polynomial of degree $d > 1$ with isolated critical points. Assume that the projective closure C of the affine curve $\{f = 0\}$ and the line at infinity $\mathbb{L}_\infty$ do not meet at d points. Then*

(i) $\mathcal{L}_\infty(f) = d - 1 - q(C, \mathbb{L}_\infty)$ *if* $\Lambda(f) = \emptyset$,
(ii) $\mathcal{L}_\infty(f) = d - 1 - \max_{t \in \Lambda(f)} q(C^t, \mathbb{L}_\infty)$ *if* $\Lambda(f) \neq \emptyset$.

Using the proposition we get

PROPERTY 3.5. *If f is a polynomial of degree $d > 1$ with isolated critical points then $\mathcal{L}_\infty(f) \geq d - 1 - q(d)$.*

PROBLEM. Compare $k(d)$ and $q(d)$ for large d. In particular, estimate $\limsup_{d\to\infty} q(d)/d^2$ like $\limsup_{d\to\infty} k(d)/d^2$ in [GZ-N].

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